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12th Sep 2025

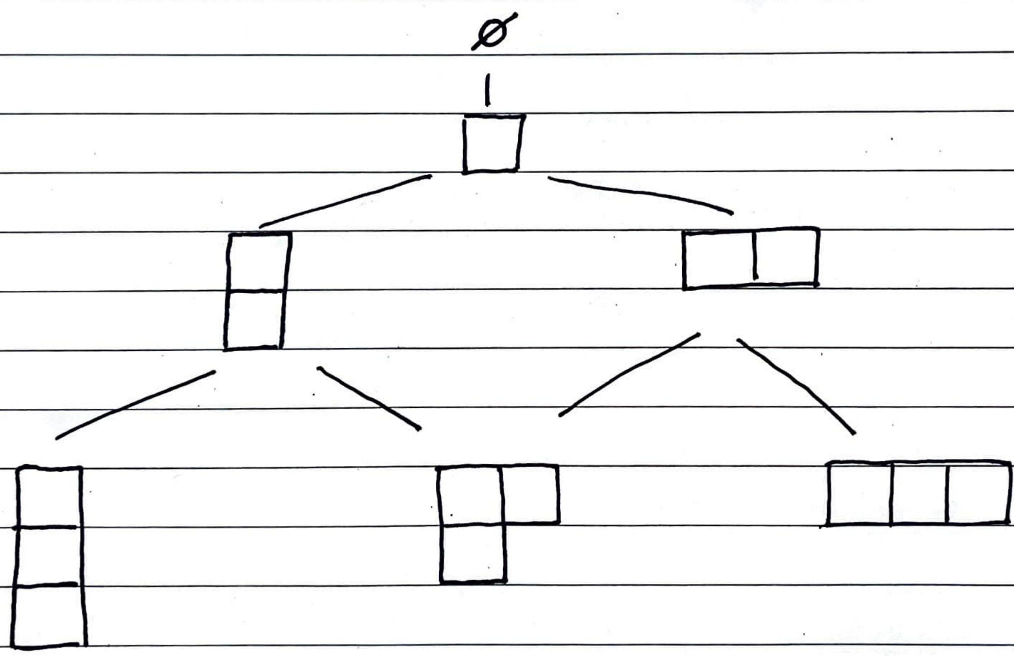
Advanced Discrete Maths
Lecture 21

Young Lattice

Recall that the Young lattice is

$$\mathbb{Y} = \bigsqcup_{k \in \mathbb{Z}_{\geq 0}} \mathbb{Y}(k)$$

where $\mathbb{Y}(k) = \{\text{partitions with } k \text{ boxes}\}$ partially ordered by inclusion.



$B(\square)$

$B(\square)^{\otimes 2}$

$B(\square)^{\otimes 3}$

The Young lattice provides the decomposition rules for when you tensor with $B(\square)$:

$$B(\square) \otimes B(\square) = B(\ydiagram{1,1}) \oplus B(\ydiagram{2})$$

$$B(\ydiagram{1,1}) \otimes B(\square) = B(\ydiagram{1,1,1}) \oplus B(\ydiagram{2,1})$$

$$B(\ydiagram{2}) \otimes B(\square) = B(\ydiagram{2,1}) \oplus B(\ydiagram{3})$$

Theorem (RSK-correspondence / Schur-Weyl Duality) Let $n, k \in \mathbb{Z}_{>0}$.

$$B(\square)^{\otimes k} \xrightarrow{\sim} \bigoplus_{\lambda \vdash k} B(\lambda) \otimes \tilde{S}_k^{\lambda}$$

where
- $B(\square)^{\otimes k} = \{$words of length k from alphabet $\{1, \dots, n\}\}$
- $B(\lambda) = \{$SSYT of shape λ filled from $\{1, \dots, n\}\}$
- $\lambda \vdash k$ means λ is a partition with k boxes.
- $\tilde{S}_k^{\lambda} = \{$standard tableaux of shape $\lambda\}$.

Defining $\tilde{f}_j$ on $B(\square)^{\otimes k}$ for $j \in \{1, \dots, n-1\}$

Let $\boxed{i_1} \otimes \cdots \boxed{i_k} \in B(\square)^{\otimes k}$.

Convert this word into a path by:

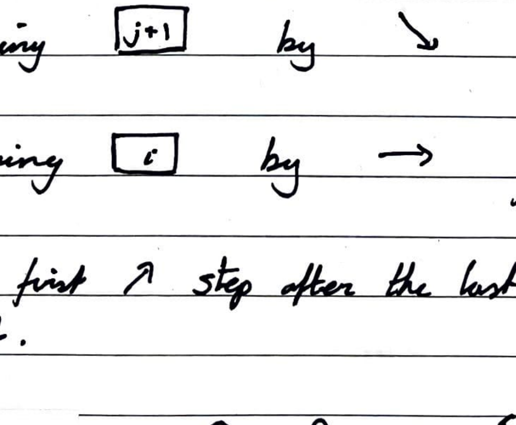
(a) Replacing $\boxed{j}$ by $\nearrow$

(b) Replacing $\boxed{j+1}$ by $\searrow$

(c) Replacing $\boxed{i}$ by $\rightarrow$ for $i \notin \{j, j+1\}$.

Let l be the first $\nearrow$ step after the last time the path reaches the minimum height.

Heuristically:



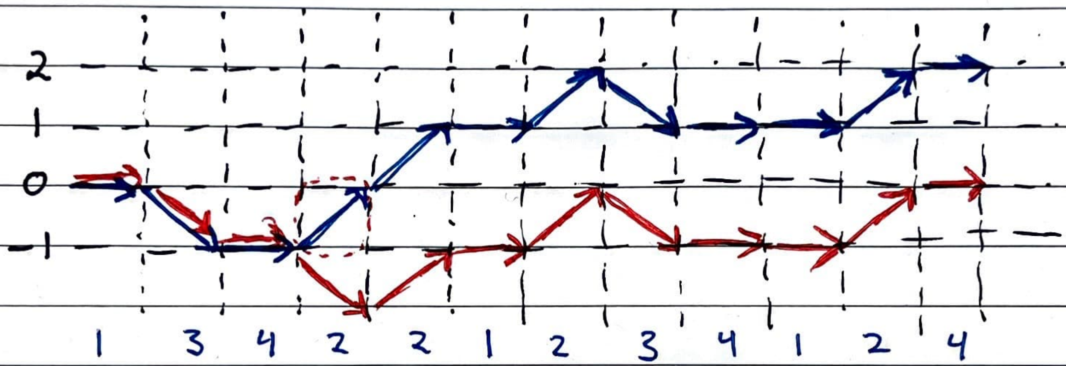
Then,

$$\tilde{f}_j\left(\boxed{i_1} \otimes \cdots \otimes \boxed{i_k}\right) = \text{same except } \boxed{i_L} \text{ is changed to } \boxed{j+1}$$

For example Suppose we wish to compute

$$\tilde{f}_2(1 \quad 3 \quad 4 \quad 2\ 2 \quad 1 \quad 2\ 3\ 4 \quad 1\ 2\ 4).$$

Here, $j=2$ and $j+1=3$ (we should keep an eye on these numbers)



Here, $L=4$ since $i_4=2$ (see red circle).
Thus,

$$\tilde{f}_2(1 \quad 3\ 4\ \underline{2}\ 2\ 1\ 2\ 3\ 4 \quad 1\ 2\ 4)$$ blue path

$$= (1 \quad 3\ 4\ \underline{3}\ 2\ 1\ 2\ 3\ 4 \quad 1\ 2\ 4)$$ red path