Gypsy Akhyar
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Advanced Discrete Mathematics
Lecture 20

There are 27 words of length 3 from the alphabet $\{1,2,3\}$.
This is the crystal $B(\square)^{\otimes 3}$ for the case $n=3$.

$$
\begin{tikzcd}
111 \arrow[r] & 211 \arrow[r] \arrow[d] & 221 \arrow[r] \arrow[d] & 222 \arrow[d] \\
 & 311 \arrow[r] & 321 \arrow[r] \arrow[d] & 322 \arrow[d] \\
 & & 331 \arrow[r] & 332 \arrow[d] \\
 & & & 333
\end{tikzcd}
$$

$$
\begin{tikzcd}
121 \arrow[r] \arrow[d] & 122 \arrow[r] & 132 \arrow[r] & 133 \arrow[d] \\
131 \arrow[r] & 231 \arrow[r] & 232 \arrow[r] & 233
\end{tikzcd}
$$

$$
\begin{tikzcd}
112 \arrow[r] \arrow[d] & 212 \arrow[r] & 312 \arrow[r] & 313 \arrow[d] \\
113 \arrow[r] & 213 \arrow[r] & 223 \arrow[r] & 323
\end{tikzcd}
$$

123

One sees here that there are 4 connected components.

Fix $n \in \mathbb{Z}_{>0}$

Recall that crystals are obtained from $B(\square) = (\boxed{1} \to \cdots \to \boxed{n})$ by taking tensor products.

<u>Example:</u> $n=3$.

$$B(\square) = 1 \xrightarrow{1} 2 \xrightarrow{2} 3$$

$$B(\square) \otimes B(\square) = \begin{tikzcd} 11 \arrow[d, "1"'] & 12 \arrow[r, "2"] & 13 \arrow[d, "1"] \\ 21 \arrow[r, "2"] \arrow[d, "2"'] & 22 \arrow[d, "2"] & 23 \\ 31 \arrow[r, "1"'] & 32 \arrow[r, "2"'] & 33 \end{tikzcd}$$

<u>Goal</u>: Decompose $B(\square)^{\otimes k}$ into <u>irreducible</u> components.

<u>Partitions</u>

A partition is $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ such that $\lambda_1 \geq \cdots \geq \lambda_n$.
A box is $(r,c) \in \mathbb{Z}^2$.
Identify a partition λ with

$$\lambda = \{(r,c) \mid r \in \{1,\dots,n\},\ c \in \{1,\dots,\lambda_r\}\}$$

Let λ be a partition and let

$$k = \lambda_1 + \cdots + \lambda_n \qquad (\text{i.e } |\lambda| = k)$$

A <u>standard tableau</u> of shape λ is a function

$$Q : \lambda \to \{1,\dots,k\}$$

satisfying:

(a) If $(r,c),(r,c+1) \in \lambda$, then $Q(r,c) < Q(r,c+1)$

(b) If $(r,c),(r+1,c) \in \lambda$, then $Q(r,c) < Q(r+1,c)$.

For example, if $\lambda = (5,3,2)$, then $k = 10$ and so an example of Q is

$Q =$

1	2	4	9	10
3	5	8		
6	7			

A semistandard Young tableau of shape λ filled from $\{1,\dots,n\}$ is a function

$$P : \lambda \to \{1,\dots,n\}$$

satisfying:

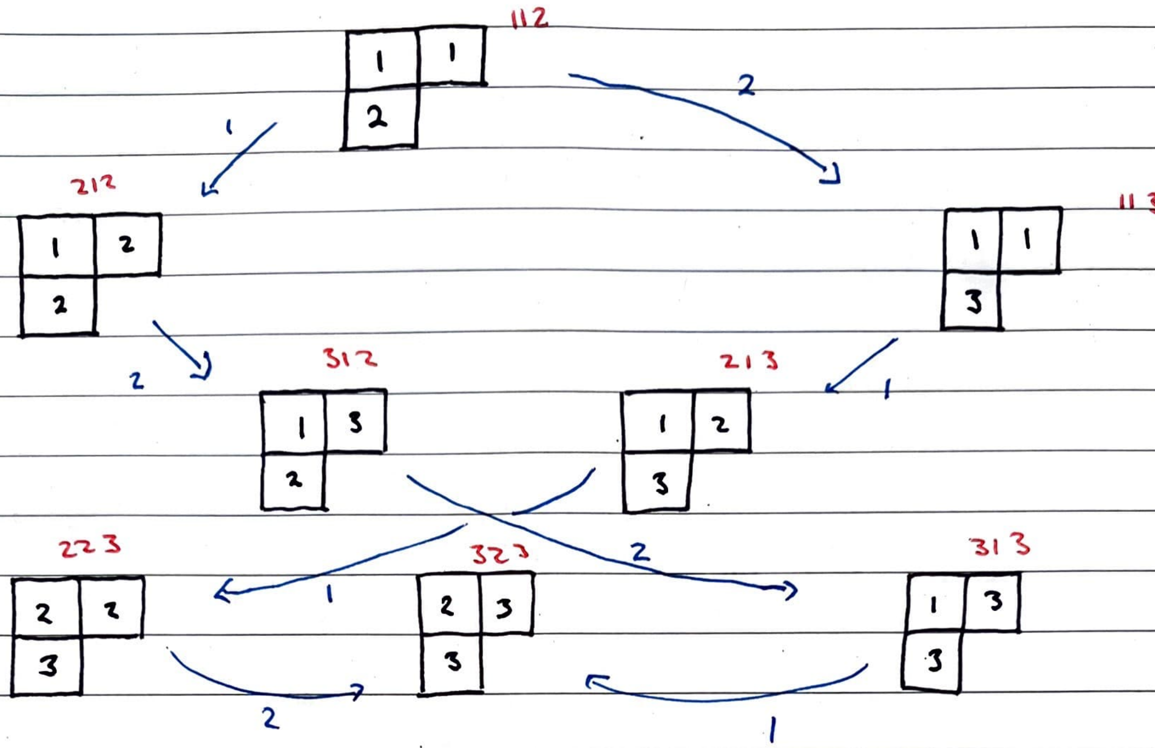
(a) If $(r,c),(r,c+1) \in \lambda$, then $P(r,c) \le P(r,c+1)$

(b) If $(r,c),(r+1,c) \in \lambda$, then $P(r,c) < P(r+1,c)$

For example

(see next page)

If $\lambda = (2,1)$ and we fill from $\{1,2\}$:



where we have indicated in red the Arabic reading word (c.f pg 1)

A *word* of length k in the alphabet $\{1,\dots,n\}$ is an element of

$$B(\square)^{\otimes k} = \{ \boxed{i_1} \otimes \cdots \otimes \boxed{i_k} \mid i_1,\dots,i_k \in \{1,\dots,n\}\}$$

The *generating function* of $B(\square)^{\otimes k}$ is

$$\begin{aligned} \mathrm{char}(B(\square)^{\otimes k}) &= \mathrm{char}(B(\square))^k \\ &= (x_1 + \cdots + x_n)^k \\ &= p_1^k \end{aligned}$$

Theorem Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with k boxes. Let Q be a standard tableau of shape λ.

Define

$$b_Q = r(Q^{-1}(1)) \otimes \cdots \otimes r(Q^{-1}(k)) \quad \in B(\square)^{\otimes k}$$

where $r(Q^{-1}(j))$ is the row number containing j in the tableau Q.

Then,

b_Q is a highest weight element of $B(\square)^{\otimes k}$.

and

all highest weight elements of $B(\square)^{\otimes k}$ are obtained this way.

Example There are 4 standard tableaux with 3 boxes

$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
$\boxed{1}\otimes\boxed{1}\otimes\boxed{1}$	$\boxed{1}\otimes\boxed{1}\otimes\boxed{2}$	$\boxed{1}\otimes\boxed{2}\otimes\boxed{1}$	$\boxed{1}\otimes\boxed{2}\otimes\boxed{3}$
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
111	112	121	123

(compare this with page 1).

Theorem Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with k boxes. Let

$$B(\lambda) = \{\text{SSYT of shape } \lambda \text{ filled from } \{1, \dots, n\}\}.$$

Then, there exists a unique crystal structure on $B(\lambda)$ such that

$$\begin{aligned} B(\lambda) &\longrightarrow B(\square)^{\otimes k} \\ P &\longmapsto \text{arabic reading word of } P \end{aligned}$$

is an injective morphism.

Theorem (RSK correspondence) There is a bijection

$$B(\square)^{\otimes k} \longleftrightarrow \left\{ (P, Q) \;\middle|\; \begin{array}{l} P \text{ is a SSYT} \\ Q \text{ is a SYT} \\ \text{shape}(P) = \text{shape}(Q) = \lambda \\ |\lambda = k \end{array} \right\}$$

$$b = \boxed{i_1} \otimes \dots \otimes \boxed{i_k} \longmapsto (P, Q)$$

where P is the standard tableau corresponding to the highest weight element in the connected component that b belongs to.

Q is the semistandard tableau corresponding to pulling back the element b to the set $B(\lambda)$ (kinda).

For example, we may rewrite the diagrams on page 1 in the following way (we do here only one connected component)

$$\begin{tikzcd}
112 \arrow[r, "1"] \arrow[d, "2"] & 212 \arrow[r, "2"] & 312 \arrow[r, "2"] & 313 \arrow[d, "1"] \\
113 \arrow[r, "1"] & 213 \arrow[r, "1"] & 223 \arrow[r, "2"] & 323
\end{tikzcd}$$

$$\updownarrow$$

$$\begin{tikzcd}[column sep=small]
\left(\ytableaushort{12,3}\,,\ \ytableaushort{11,2}\right) \arrow[r, "1"] \arrow[d, "2"] & \left(\ytableaushort{12,3}\,,\ \ytableaushort{12,2}\right) \arrow[r, "2"] & \left(\ytableaushort{12,3}\,,\ \ytableaushort{13,2}\right) \arrow[r, "2"] & \left(\ytableaushort{12,3}\,,\ \ytableaushort{13,3}\right) \arrow[d, "1"] \\
\left(\ytableaushort{12,3}\,,\ \ytableaushort{11,3}\right) \arrow[r, "1"] & \left(\ytableaushort{12,3}\,,\ \ytableaushort{12,3}\right) \arrow[r, "1"] & \left(\ytableaushort{12,3}\,,\ \ytableaushort{22,3}\right) \arrow[r, "2"] & \left(\ytableaushort{12,3}\,,\ \ytableaushort{23,3}\right)
\end{tikzcd}$$

(c.f. pp 4 and 5).

Remark If $\lambda = (\lambda_1, \dots, \lambda_n)$ has n boxes (i.e. if $k = n$), then every standard tableau Q is a semi-standard tableau filled from $\{1, \dots, n\}$. In this way, we got a bijection:

$$S_n = \left\{ \begin{array}{c} \text{words with each letter appearing} \\ \text{once} \end{array} \right\} \longleftrightarrow \{ \text{pairs } (P, Q) \text{ of SYT} \}$$

Taking cardinalities gives

$$n! = \sum_{\lambda \text{ with } n \text{ boxes}} f_\lambda^2$$

where $f_\lambda = \#$ standard tableau of shape λ.