Advanced Discrete Mathematics
Lecture 19

Crystals

Data : $(B, wt, \tilde{f}_1, \ldots, \tilde{f}_{n-1})$, $\left(\begin{array}{l} B \text{ is a set} \\ wt : B \to \mathbb{Z}^n \\ \tilde{f}_i \text{ are } \{0,1\}\text{-matrices} \end{array} \right)$

Let $n \in \mathbb{Z}_{>0}$

$$\underline{B(\square)} = \underset{\varepsilon_1}{\boxed{1}} \xrightarrow{\tilde{f}_1} \underset{\varepsilon_2}{\boxed{2}} \xrightarrow{\tilde{f}_2} \cdots \xrightarrow{\tilde{f}_{n-1}} \underset{\varepsilon_n}{\boxed{n}}$$

where the red indicates the weight of that element and

$$\varepsilon_i = (0, \ldots, 0, \overset{i^{\text{th}} \text{ spot}}{1}, 0, \ldots, 0) \in \mathbb{Z}^n.$$

Every crystal is generated from tensor products:

$$B(\square) \otimes B(\square) \otimes \cdots$$

The tensor product of $B_1 = (B_1, wt^{(1)}, \tilde{f}_1^{(1)}, \ldots, \tilde{f}_{n-1}^{(1)})$ and $B_2 = (B_2, wt^{(2)}, \tilde{f}_1^{(2)}, \ldots, \tilde{f}_{n-1}^{(2)})$ is

$$\underline{B_1 \otimes B_2} = (B_1 \times B_2, wt, \tilde{f}_1, \ldots, \tilde{f}_{n-1}) \text{ given by}$$

$$wt(b_1 \otimes b_2) = wt^{(1)}(b_1) + wt^{(2)}(b_2)$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i^{(1)} b_1 \otimes b_2, & \text{if } d_i^+(b_1) > d_i^-(b_2) \\ b_1 \otimes \tilde{f}_i^{(2)} b_2, & \text{if } d_i^+(b_1) \leq d_i^-(b_2) \end{cases}$$

For <u>example</u>, if $n=3$,

$$B(\square) = \boxed{1} \xrightarrow{\tilde{f}_1} \boxed{2} \xrightarrow{\tilde{f}_2} \boxed{3}$$

$$B(\square)\otimes B(\square) = \begin{tikzcd}
\boxed{1}\otimes\boxed{1} \arrow[d, "\tilde{f}_1"'] & \boxed{1}\otimes\boxed{2} \arrow[r, "\tilde{f}_2"] & \boxed{1}\otimes\boxed{3} \arrow[d, "\tilde{f}_1"] \\
\boxed{2}\otimes\boxed{1} \arrow[r, "\tilde{f}_2"] \arrow[d, "\tilde{f}_2"'] & \boxed{2}\otimes\boxed{2} \arrow[d, "\tilde{f}_2"'] & \boxed{2}\otimes\boxed{3} \\
\boxed{3}\otimes\boxed{1} \arrow[r, "\tilde{f}_1"'] & \boxed{3}\otimes\boxed{2} \arrow[r, "\tilde{f}_2"'] & \boxed{3}\otimes\boxed{3}
\end{tikzcd}$$

$$= \begin{tikzcd}
11 \arrow[d, "\tilde{f}_1"'] & 12 \arrow[r, "\tilde{f}_2"] & 13 \arrow[d, "\tilde{f}_1"] \\
21 \arrow[r, "\tilde{f}_1"] \arrow[d, "\tilde{f}_2"'] & 22 \arrow[d, "\tilde{f}_2"] & 23 \\
31 \arrow[r, "\tilde{f}_1"'] & 32 \arrow[r, "\tilde{f}_2"'] & 33
\end{tikzcd}$$

Notice that there are two connected components.

Let B_1 and B_2 be crystals.

A <u>morphism</u> from B_1 to B_2 is a function $\Phi : B_1 \to B_2$ such that

(a) if $b_1 \in B_1$, then $\mathrm{wt}(\Phi(b_1)) = \mathrm{wt}(b_1)$

(b) if $i \in \{1, \dots, n-1\}$ and $b_1 \in B_1$ then $\Phi(\tilde{f}_i^{(1)} b_1) = \tilde{f}_i^{(2)} \Phi(b_1)$.

Property (a) tells us to respect weights and property (b) tells us to respect the crystal operators.

An <u>isomorphism</u> of crystals is a morphism $\Phi : B_1 \to B_2$ such that there exists a crystal morphism $\Psi : B_2 \to B_1$ with

$$\Psi \circ \Phi = id_{B_1} \qquad \text{and} \qquad \Phi \circ \Psi = id_{B_2}.$$

A <u>subcrystal</u> of B is a subset of B that is closed under $\tilde{e}_1, \ldots, \tilde{e}_{n-1}$ and $\tilde{f}_1, \ldots, \tilde{f}_{n-1}$.

An <u>irreducible</u> or <u>simple</u> crystal is a crystal B with no subcrystals except $\emptyset$ and B.

The two subcrystals of $B(\square) \otimes B(\square)$ are

$$B(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{\tilde{f}_2} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \xrightarrow{\tilde{f}_1} \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}$$

$$B(\square\square) = \begin{tikzcd} \begin{array}{|c|c|}\hline 1 & 1\\\hline\end{array} \arrow[r, "\tilde{f}_1"] & \begin{array}{|c|c|}\hline 1 & 2\\\hline\end{array} \arrow[r, "\tilde{f}_2"] \arrow[d, "\tilde{f}_1"'] & \begin{array}{|c|c|}\hline 1 & 3\\\hline\end{array} \arrow[d, "\tilde{f}_1"] \\ & \begin{array}{|c|c|}\hline 2 & 2\\\hline\end{array} \arrow[r, "\tilde{f}_2"] & \begin{array}{|c|c|}\hline 2 & 3\\\hline\end{array} \arrow[d, "\tilde{f}_2"] \\ & & \begin{array}{|c|c|}\hline 3 & 3\\\hline\end{array} \end{tikzcd}$$

In this case, we write

$$B(\square) \otimes B(\square) \simeq B(\begin{smallmatrix}\square\\\square\end{smallmatrix}) \oplus B(\square\square)$$

More precisely, if B_1 and B_2 are crystals, then

$$\underline{B_1 \oplus B_2} = (B_1 \sqcup B_2, \mathrm{wt}, \tilde{f}_1, \dots, \tilde{f}_{n-1})$$

given by, if $b \in B_1 \sqcup B_2$, then

$$\mathrm{wt}(b) = \begin{cases} \mathrm{wt}^{(1)}(b), & \text{if } b \in B_1 \\ \mathrm{wt}^{(2)}(b), & \text{if } b \in B_2 \end{cases}$$

and

$$\tilde{f}_i b = \begin{cases} \tilde{f}_i^{(1)}(b), & \text{if } b \in B_1 \\ \tilde{f}_i^{(2)}(b), & \text{if } b \in B_2 \end{cases}$$

$B_1 \oplus B_2$ is called the <u>direct sum</u> of B_1 and B_2.

A <u>highest weight element</u> in a crystal B is $b \in B$ such that

if $i \in \{1, \dots, n-1\}$, then $\tilde{e}_i b = 0$

Define $\underline{B^+} = \{\text{highest weight elements in } B\}$

<u>Theorem</u> Let B be a crystal.

(a) B is irreducible if and only if the crystal graph of B is connected.

(b) B is irreducible if and only if $\mathrm{Card}(B^+) = 1$.

The character of B is the weight generating function of B:

$$\mathrm{char}(B) = \sum_{p \in B} x^{\mathrm{wt}(p)}.$$

where if $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, then $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$.

Proposition Let B_1 and B_2 be crystals.

$$\mathrm{char}(B_1 \oplus B_2) = \mathrm{char}(B_1) + \mathrm{char}(B_2)$$

$$\mathrm{char}(B_1 \otimes B_2) = \mathrm{char}(B_1) \cdot \mathrm{char}(B_2).$$