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Advanced Discrete Mathematics

Lecture 18

Crystals

Example 1

$$\boxed{1} \xrightarrow{\tilde{f}_1} \boxed{2} \xrightarrow{\tilde{f}_2} \boxed{3} = B(\square)$$

Example 2

$$\begin{array}{ccccc} \boxed{1} \boxed{1} & \xrightarrow{\tilde{f}_1} & \boxed{1} \boxed{2} & \xrightarrow{\tilde{f}_2} & \boxed{1} \boxed{3} \\ & & \downarrow \tilde{f}_1 & & \downarrow \tilde{f}_1 \\ & & \boxed{2} \boxed{2} & \xrightarrow{\tilde{f}_2} & \boxed{2} \boxed{3} = B(\square) \\ & & & & \downarrow \tilde{f}_2 \\ & & & & \boxed{3} \boxed{3} \end{array}$$

Example 3

$$\boxed{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}} \xrightarrow{\tilde{f}_2} \boxed{\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}} \xrightarrow{\tilde{f}_1} \boxed{\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}} = B(\square)$$

Example 4

$$\begin{array}{ccccccc} & \tilde{f}_1 \nearrow & \boxed{\begin{smallmatrix} 1 & 2 \\ 2 & \end{smallmatrix}} & \xrightarrow{\tilde{f}_2} & \boxed{\begin{smallmatrix} 1 & 3 \\ 2 & \end{smallmatrix}} & \xrightarrow{\tilde{f}_2} & \boxed{\begin{smallmatrix} 2 & 2 \\ 3 & \end{smallmatrix}} & \xrightarrow{\tilde{f}_2} & \boxed{\begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix}} \\ \boxed{\begin{smallmatrix} 1 & 1 \\ 2 & \end{smallmatrix}} & \searrow \tilde{f}_2 & \boxed{\begin{smallmatrix} 1 & 1 \\ 3 & \end{smallmatrix}} & \xrightarrow{\tilde{f}_1} & \boxed{\begin{smallmatrix} 1 & 2 \\ 3 & \end{smallmatrix}} & \xrightarrow{\tilde{f}_2} & \boxed{\begin{smallmatrix} 1 & 3 \\ 3 & \end{smallmatrix}} & \xrightarrow{\tilde{f}_2} & \boxed{\begin{smallmatrix} 2 & 3 \\ 3 & \end{smallmatrix}} \end{array}$$

Example (characters)

Looking at the examples on the previous page:

$$\begin{aligned} \bullet \text{ char}(B(\square)) &= x_1 + x_2 + x_3 = e_1(x_1, x_2, x_3) \\ &= h_1(x_1, x_2, x_3) \end{aligned}$$

$$\begin{aligned} \bullet \text{ char}(B(\square\square)) &= x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2 \\ &= h_2(x_1, x_2, x_3) \end{aligned}$$

$$\bullet \text{ char}(B(\boxplus)) = x_1x_2 + x_1x_3 + x_2x_3 = e_2(x_1, x_2, x_3)$$

$$\begin{aligned} \bullet \text{ char}(B(\boxtimes)) &= x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3 \\ &\quad + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \\ &= S_{(2,1)}(x_1, x_2, x_3) \end{aligned}$$

One may think of the character as the generating function of the crystal.

A crystal is an object in the category of crystals.

The category of crystals is the (full Karasubian envelope) subcategory of WB-crystals generated by $B(\square)$ (under tensor products and idempotent projections).

(Arun said to ignore the stuff in red.)

Roots and simple roots

Let $\epsilon_1, \dots, \epsilon_n \in \mathbb{Z}^n$ be given by $\epsilon_i = (0, \dots, 0, \overset{i\text{th}}{1}, 0, \dots, 0)$

The set of roots is $R = \{\epsilon_i - \epsilon_j \mid i, j \in \{1, \dots, n\} \text{ and } i \neq j\}$.

The simple roots are $\alpha_1, \dots, \alpha_{n-1}$ given by

$$\alpha_i = \epsilon_i - \epsilon_{i+1}, \quad \text{for } i \in \{1, \dots, n-1\}.$$

A framework is a set B with a function $\text{wt}: B \rightarrow \mathbb{Z}^n$

A WB-crystal is a framework (B, wt) with

$$\tilde{f}_1, \tilde{f}_2, \dots, \tilde{f}_{n-1} \in M_{B \times B}(\{0, 1\})$$

such that

(a) $\tilde{f}_i$ has at most one 1 in each column.

(b) If $(\tilde{f}_i)_{b'b} = 1$ then $\text{wt}(b') = \text{wt}(b) - \alpha_i$.

In this way, we have

$$\tilde{f}_i b = \begin{cases} b', & \text{if } (\tilde{f}_i)_{b'b} = 1 \\ 0, & \text{if } (\tilde{f}_i)_{b'b} = 0 \end{cases}, \text{ for } b' \in B$$

In other words, (or pictures)

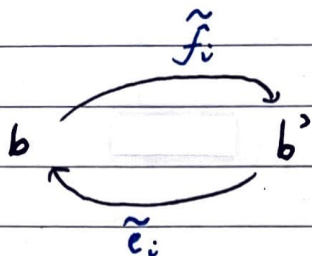
$$b \xrightarrow{\tilde{f}_i} b'$$

i-strings Let $(B, \text{wt}, \tilde{f}_1, \dots, \tilde{f}_{n-1})$ be a WB-crystal.

Define

$\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n$ by $e_i = \tilde{f}_i^T$ for $i \in \{1, \dots, n-1\}$.

Pictorially:



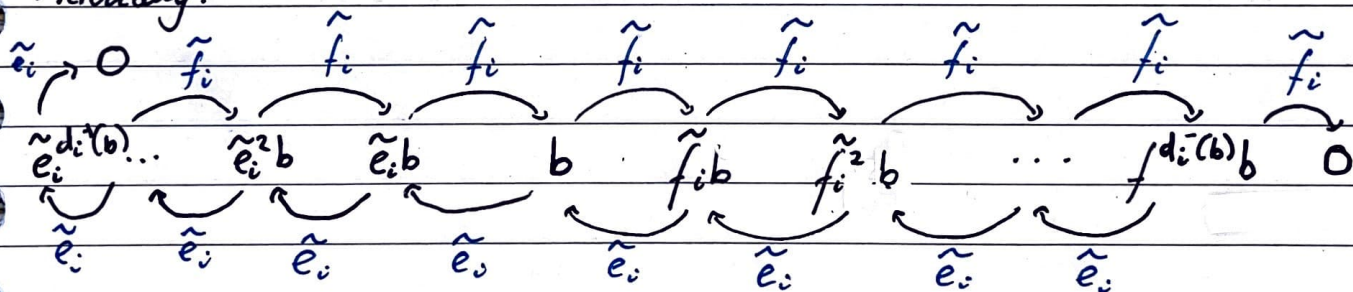
Let $b \in B$

Define $d_i^-(b), d_i^+(b) \in \mathbb{Z}_{\geq 0}$ such that

$$\tilde{e}_i^{d_i^+(b)} b \neq 0 \quad \text{and} \quad \tilde{e}_i^{d_i^+(b)+1} b = 0$$

$$\tilde{f}_i^{d_i^-(b)} b \neq 0 \quad \text{and} \quad \tilde{f}_i^{d_i^-(b)+1} b = 0$$

Pictorially:



The i-string of b is the set

$$S_i(b) = \{ \underset{\substack{\uparrow \\ \text{head}}}{\tilde{e}_i^{d_i^+(b)} b}, \dots, \tilde{e}_i b, b, \tilde{f}_i b, \dots, \underset{\substack{\uparrow \\ \text{tail}}}{\tilde{f}_i^{d_i^-(b)} b} \}$$

Tensor product

Let $B_1 = (B_1, \text{wt}^{(1)}, \tilde{f}_1^{(1)}, \dots, \tilde{f}_{n-1}^{(1)})$ and

$$B_2 = (B_2, \text{wt}^{(2)}, \tilde{f}_1^{(2)}, \dots, \tilde{f}_{n-1}^{(2)})$$

be WB-crystals. The tensor product of B_1 and B_2 is

$$B_1 \otimes B_2 = (B_1 \times B_2, \text{wt}, \tilde{f}_1, \dots, \tilde{f}_{n-1})$$

where we write $b_1 \otimes b_2 \in B_1 \times B_2$ instead of $(b_1, b_2) \in B_1 \times B_2$ such that

$$\text{wt}(b_1 \otimes b_2) = \text{wt}^{(1)}(b_1) + \text{wt}^{(2)}(b_2)$$

and

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i^{(1)} b_1 \otimes b_2, & \text{if } d_i^+(b_1) > d_i^-(b_2) \\ b_1 \otimes \tilde{f}_i^{(2)} b_2, & \text{if } d_i^+(b_1) \leq d_i^-(b_2) \end{cases}$$

Theorem (Monoidal Structure)

$$\tilde{f}_i((b_1 \otimes b_2) \otimes b_3) = \tilde{f}_i(b_1 \otimes (b_2 \otimes b_3)).$$