Gypsy Akhyar
5th Sep 2025

Advanced Discrete Mathematics
Lecture 18

Crystals

Example 1

$$\boxed{1} \xrightarrow{\tilde{f}_1} \boxed{2} \xrightarrow{\tilde{f}_2} \boxed{3} = B(\square)$$

Example 2

$$\begin{tikzcd}
\begin{array}{|c|c|}\hline 1&1\\\hline\end{array} \arrow[r, "\tilde{f}_1"] & \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \arrow[r, "\tilde{f}_2"] \arrow[d, "\tilde{f}_1"] & \begin{array}{|c|c|}\hline 1&3\\\hline\end{array} \arrow[d, "\tilde{f}_1"] \\
& \begin{array}{|c|c|}\hline 2&2\\\hline\end{array} \arrow[r, "\tilde{f}_2"] & \begin{array}{|c|c|}\hline 2&3\\\hline\end{array} \arrow[d, "\tilde{f}_2"] & = B(\square\square) \\
& & \begin{array}{|c|c|}\hline 3&3\\\hline\end{array}
\end{tikzcd}$$

Example 3

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \xrightarrow{\tilde{f}_2} \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} \xrightarrow{\tilde{f}_1} \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = B\left(\begin{array}{|c|}\hline \\\hline \\\hline\end{array}\right)$$

Example 4

$$\begin{tikzcd}
& \begin{array}{|c|c|}\hline 1&2\\\hline 2\\\cline{1-1}\end{array} \arrow[r, "\tilde{f}_2"] & \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \arrow[ddr, "\tilde{f}_2"] & \begin{array}{|c|c|}\hline 2&2\\\hline 3\\\cline{1-1}\end{array} \arrow[dr, "\tilde{f}_2"] & \\
\begin{array}{|c|c|}\hline 1&1\\\hline 2\\\cline{1-1}\end{array} \arrow[ur, "\tilde{f}_1"] \arrow[dr, "\tilde{f}_2"'] & & & & \begin{array}{|c|c|}\hline 2&3\\\hline 3\\\cline{1-1}\end{array} \\
& \begin{array}{|c|c|}\hline 1&1\\\hline 3\\\cline{1-1}\end{array} \arrow[r, "\tilde{f}_1"] & \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \arrow[uur, "\tilde{f}_2"'] & \begin{array}{|c|c|}\hline 1&3\\\hline 3\\\cline{1-1}\end{array} \arrow[ur, "\tilde{f}_2"'] &
\end{tikzcd}$$

Example (characters)

Looking at the examples on the previous page:

- $\mathrm{char}(B(\square)) = x_1 + x_2 + x_3 = e_1(x_1, x_2, x_3)$
  $= h_1(x_1, x_2, x_3)$

- $\mathrm{char}(B(\square\square)) = x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2$
  $= h_2(x_1, x_2, x_3)$

- $\mathrm{char}(B(\boxminus)) = x_1x_2 + x_1x_3 + x_2x_3 = e_2(x_1, x_2, x_3)$

- $\mathrm{char}(B(\boxplus)) = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 2x_1x_2x_3$
  $+ x_1x_3^2 + x_2^2x_3 + x_2x_3^2$
  $= S_{(2,1)}(x_1, x_2, x_3)$

One may think of the character as the generating function of the crystal.

A crystal is an object in the category of crystals:

The category of crystals is the (full Karoubian envelope) subcategory of WB-crystals generated by $B(\square)$ (under tensor products and idempotent projections).

(Arun said to ignore the stuff in red)

Roots and simple roots

Let $\varepsilon_1, \dots, \varepsilon_n \in \mathbb{Z}^n$ be given by $\varepsilon_i = (0, \dots, 0, \overset{i\text{th}}{1}, 0, \dots, 0)$

The set of roots is $R = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \ldots, n\}$ and $i \neq j\}$.

The simple roots are $\alpha_1, \ldots, \alpha_{n-1}$ given by

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \qquad \text{for} \quad i \in \{1, \ldots, n-1\}.$$

A framework is a set B with a function $wt : B \to \mathbb{Z}^n$

A WB-crystal is a framework (B, wt) with

$$\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_{n-1} \in M_{B \times B}(\{0,1\})$$

such that

(a) $\tilde{f}_i$ has at most one 1 in each column.

(b) If $(\tilde{f}_i)_{b'b} = 1$ then $wt(b') = wt(b) - \alpha_i$.

In this way, we have

$$\tilde{f}_i b = \begin{cases} b', & \text{if } (\tilde{f}_i)_{b'b} = 1 \\ 0, & \text{if } (\tilde{f}_i)_{b'b} = 0 \end{cases}, \quad \text{for } b' \in B$$

In other words, (or pictures)

$$b \xrightarrow{\tilde{f}_i} b'$$

i-strings Let $(B, wt, \tilde{f}_1, \ldots, \tilde{f}_{n-1})$ be a WB-crystal.

Define

$\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_{n-1}$ by $e_i = \tilde{f}_i^{T}$ for $i \in \{1, \dots, n-1\}$.

Pictorially:

$$\begin{tikzcd} b \arrow[r, bend left, "\tilde{f}_i"] & b' \arrow[l, bend left, "\tilde{e}_i"] \end{tikzcd}$$

Let $b \in B$

Define $d_i^-(b), d_i^+(b) \in \mathbb{Z}_{\geq 0}$ such that

$$\tilde{e}_i^{\,d_i^+(b)} b \neq 0 \quad \text{and} \quad \tilde{e}_i^{\,d_i^+(b)+1} b = 0$$

$$\tilde{f}_i^{\,d_i^-(b)} b \neq 0 \quad \text{and} \quad \tilde{f}_i^{\,d_i^-(b)+1} b = 0$$

Pictorially:

$$\begin{tikzcd}[column sep=small]
0 & & & & & & & & & \\
\tilde{e}_i^{\,d_i^+(b)} \cdots \arrow[u, "\tilde{e}_i"] \arrow[loop below, "\tilde{e}_i"] \arrow[r, bend left, "\tilde{f}_i"] & \tilde{e}_i^2 b \arrow[l, bend left, "\tilde{e}_i"] \arrow[r, bend left, "\tilde{f}_i"] & \tilde{e}_i b \arrow[l, bend left, "\tilde{e}_i"] \arrow[r, bend left, "\tilde{f}_i"] & b \arrow[l, bend left, "\tilde{e}_i"] \arrow[r, bend left, "\tilde{f}_i"] & \tilde{f}_i b \arrow[l, bend left, "\tilde{e}_i"] \arrow[r, bend left, "\tilde{f}_i"] & \tilde{f}_i^2 b \arrow[l, bend left, "\tilde{e}_i"] \arrow[r, bend left, "\tilde{f}_i"] & \cdots \arrow[l, bend left, "\tilde{e}_i"] \arrow[r, bend left, "\tilde{f}_i"] & \tilde{f}_i^{\,d_i^-(b)} b \arrow[l, bend left, "\tilde{e}_i"] \arrow[r, bend left, "\tilde{f}_i"] & 0
\end{tikzcd}$$

The *i-string of b* is the set

$$S_i(b) = \{ \underset{\text{head}}{\tilde{e}_i^{\,d_i^+(b)}}, \dots, \tilde{e}_i b, b, \tilde{f}_i b, \dots, \underset{\text{tail}}{\tilde{f}_i^{\,d_i^-(b)}} \}$$

Tensor product

Let $B_1 = (B_1, wt^{(1)}, \tilde{f}_1^{(1)}, \dots, \tilde{f}_{n-1}^{(1)})$ and

$B_2 = (B_2, wt^{(2)}, \tilde{f}_1^{(2)}, \dots, \tilde{f}_{n-1}^{(2)})$

be WB-crystals. The <u>tensor product of B_1 and B_2</u> is

$$B_1 \otimes B_2 = (B_1 \times B_2, wt, \tilde{f}_1, \dots, \tilde{f}_{n-1})$$

where we write $b_1 \otimes b_2 \in B_1 \times B_2$ instead of $(b_1, b_2) \in B_1 \times B_2$ such that

$$wt(b_1 \otimes b_2) = wt^{(1)}(b_1) + wt^{(2)}(b_2)$$

and

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i^{(1)} b_1 \otimes b_2, & \text{if } d_i^+(b_1) > d_i^-(b_2) \\ b_1 \otimes \tilde{f}_i^{(2)} b_2, & \text{if } d_i^+(b_1) \leq d_i^-(b_2) \end{cases}$$

<u>Theorem (Monoidal Structure)</u>

$$\tilde{f}_i((b_1 \otimes b_2) \otimes b_3) = \tilde{f}_i(b_1 \otimes (b_2 \otimes b_3)).$$