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4th Sep 2025

Advanced Discrete Mathematics
Problem Solving 17

Generalised cohomology theories

Examples:

H^*, ordinary cohomology $\qquad x \oplus y = x + y$

K, K-theory $\qquad x \oplus y = x + y + xy$

Elliptic cohomology

$\vdots$

Morava K-theory

$\vdots$

Ω, cobordism

Somehow these are related to formal groups with a formal group law

$$x \oplus y = x + y + a_{11} xy + \cdots \quad \text{(a formal power series)}$$

that satisfies some relations (perhaps that of an abelian group)

Vaguely, a cohomology theory h is a functor

$$f : \{\text{spaces}\} \longrightarrow \{\text{(graded) commutative rings}\}$$

This is related to symmetric functions as we have a correspondence:

power operators $\longleftrightarrow$ power sum symmetric functions

Euler classes $\longleftrightarrow$ elementary symmetric functions

<u>Quantum cohomology</u> (a priori not a generalised cohomology theory)

From X (a space) build $LX = \{f: S^1 \to X\}$
$PX = \{f: \mathbb{R}_{[0,1]} \to X\}$

quantum cohomology

$QH(X)$ is vaguely $H(LX)$ or $H(PX)$

$QK(X)$ is vaguely $K(LX)$ or $K(PX)$

In this way, you get a quantum version of each generalised cohomology theory.

Another link to symmetric functions:

H(grassmannians) = symmetric functions with ordinary multiplication

$$S_\lambda S_\mu = \sum_\nu c^\nu_{\lambda\mu} S_\nu \quad \text{where} \quad c^\nu_{\lambda\mu} \in \mathbb{Z}_{\geq 0}$$

and

QH(grassmannians) = symmetric functions with <u>deformed</u> multiplication

$$S_\lambda * S_\mu = \sum_\nu c^\nu_{\lambda\mu}(q) S_\nu \quad \text{where} \quad c^\nu_{\lambda\mu}(q) \in \mathbb{Z}[q]$$

where there is a specialisation of q that gives $c^\nu_{\lambda\mu}(q) \xrightarrow{q=?} c^\nu_{\lambda\mu}$.
These are <u>Gromov-Witten invariants</u>.

In particular, we have the following connections:

symmetric functions
and
$$s_\lambda s_\mu = \sum_\nu c_{\lambda\mu}^\nu s_\nu$$

we know

we know

H(Grassmannians) Representations of $GL_n(\mathbb{C})$

Borel-Weil

and

deformed symmetric functions
and
$$s_\lambda * s_\mu = \sum_\nu c_{\lambda\mu}^\nu(q)\, s_\nu$$

we know

Braverman-Finkelberg-Gaitsgory

QH(Grassmannian) Representations of ???

(Maybe Borel-Weil on semi-infinite flag varieties)

(maybe related to level 0 representations of quantum affine algebras)