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Advanced Discrete Mathematics
Lecture 17

Binomial Theorem

Positive version if $n \in \mathbb{Z}_{\geq 0}$ then

$$(u+z)^n = \sum_{r=0}^{n} u^{n-r} z^r \binom{n}{r}$$

Negative version if $n \in \mathbb{Z}_{\geq 0}$ then

$$(u+z)^{-n} = \sum_{r \in \mathbb{Z}_{\geq 0}} u^{-n-r} z^r \binom{n+r-1}{r}$$

how do we make sense of this?

To make sense of these binomial coefficients

$$\binom{1/3}{7}, \quad \binom{-\pi}{6}, \quad \text{etc.}$$

we turn to calculus and Taylor series:

$$(1+x)^\alpha = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{1}{k!} \left(\frac{d^k}{dx^k} f \right) \Big|_{x=0} x^k,$$

where $f(x) = (1+x)^\alpha$.

Notice that:
$$\frac{d}{dx}(1+x)^\alpha = \alpha (1+x)^{\alpha-1}$$
$$\frac{d^2}{dx^2}(1+x)^\alpha = \alpha(\alpha-1)(1+x)^{\alpha-2}$$
$$\frac{d^3}{dx^3}(1+x)^\alpha = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$$
$$\vdots$$

So this gives $\frac{1}{1!}\frac{d}{dx}(1+x)^{\alpha} = \alpha \cdot (1+x)^{\alpha}$

$$\frac{1}{2!}\frac{d^2}{dx^2}(1+x)^{\alpha} = \frac{\alpha(\alpha-1)}{2\cdot 1} (1+x)^{\alpha}$$

$$\frac{1}{3!}\frac{d^3}{dx^3}(1+x)^{\alpha} = \frac{\alpha(\alpha-1)(\alpha-2)}{3\cdot 2\cdot 1} (1+x)^{\alpha}.$$

In this way, we ought define

$$\binom{\alpha}{r} = \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}$$

The positive binomial theorem is the $x_1 = x_2 = \cdots = x_n = 1$ specialisation of the definition of the elementary symmetric function. That is, recall

$$\prod_{i=1}^{n} (u + x_i z) = \prod_{i=1}^{n} u(1+u^{-1}x_i z)$$

$$= u^n \prod_{i=1}^{n} (1+u^{-1}x_i z)$$

$$= u^n \sum_{r=0}^{n} e_r(u^{-1}x_1, \dots, u^{-1}x_n)\, z^r$$

$$= \sum_{r=0}^{n} e_r(x_1, \dots, x_n)\, u^{n-r} z^r$$

Yet, we have shown $e_r(x_1, \dots, x_n) = \sum_{1 \le i_1 < \cdots < i_r \le n} x_{i_1} \cdots x_{i_r}$

Notice that

$$e_r(1,\dots,1) = \sum_{1\le i_1<\dots<i_r\le n} 1$$

$$= \#\ r\text{-subsets of } \{1,\dots,n\}$$

$$= \binom{n}{r}$$

So,

$$\prod_{i=1}^{n}(u+x_i z) = \sum_{r=0}^{n} e_r(x_1,\dots,x_n)\, u^{n-r} z^r$$

gives, at $x_1 = x_2 = \dots = x_n = 1$,

$$(u+z)^n = \sum_{r=0}^{n}\binom{n}{r} u^{n-r} z^r .$$

The negative binomial theorem is the $x_1 = x_2 = \dots = x_n = 1$ specialisation of the definition of the homogeneous symmetric function. That is, recall

$$\prod_{i=1}^{n}\frac{1}{u - x_i z} = u^{-n}\prod_{i=1}^{n}\frac{1}{1-u^{-1}x_i z}$$

$$= u^{-n}\sum_{r\in\mathbb{Z}_{\ge 0}} h_r(u^{-1}x_1,\dots,u^{-1}x_n)\, z^r$$

$$= \sum_{r\in\mathbb{Z}_{\ge 0}} h_r(x_1,\dots,x_n)\, u^{-n-r} z^r$$

Yet, we have shown $h_r(x_1,\dots,x_n) = \sum_{1\le i_1\le\cdots\le i_r\le n} x_{i_1}\cdots x_{i_r}$.

Notice that

$$h_r(1,\dots,1) = \sum_{1\le i_1\le\cdots\le i_r\le n} 1$$

$$= \#\text{weakly increasing } r\text{-sequences of } \{1,\dots,n\}$$

$$= \binom{n+r-1}{r}$$

So,

$$\prod_{i=1}^{n} \frac{1}{u-x_i z} = \sum_{r\in\mathbb{Z}_{\ge 0}} h_r(x_1,\dots,x_n)\, u^{-n-r} z^r$$

gives, at $x_1 = x_2 = \cdots = x_n = 1$,

$$(u-z)^{-n} = \sum_{r\in\mathbb{Z}_{\ge0}} \binom{n+r-1}{r} u^{-n-r} z^r$$

<u>Finite q-binomial theorems</u> if $n\in\mathbb{Z}_{>0}$ then

$$\prod_{i=1}^{n}(u+q^{i-1}z) = \sum_{r=0}^{n} q^{\frac{1}{2}r(r-1)} \begin{bmatrix} n \\ r \end{bmatrix} u^{n-r} z^r$$

where $\begin{bmatrix} n \\ r \end{bmatrix} = \mathrm{Card}(Gr(\mathbb{F}_q{}^n))$

and

$$\prod_{i=1}^{n} \frac{1}{u-q^{i-1}z} = \sum_{r\in\mathbb{Z}_{\geq 0}} \begin{bmatrix} n+r-1 \\ r \end{bmatrix} u^{n-r} z^{r}$$

where $\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}$, $[r]! = [r][r-1]\cdots[2][1]$, $[r] = \frac{q^r-1}{q-1}$.

These are the specialisations at

$$x_1 = 1, \quad x_2 = q, \quad x_3 = q^2, \quad \dots \quad (\text{i.e. } x_i = q^{i-1})$$

This is called the principal specialisation.

So

$$e_r(1, q, q^2, \dots, q^{n-1}) = \begin{bmatrix} n \\ r \end{bmatrix}$$

$$h_r(1, q, q^2, \dots, q^{n-1}) = \begin{bmatrix} n+r-1 \\ r \end{bmatrix}$$

The infinite q-binomial theorem is

$$\frac{(tz;q)_\infty}{(uz;q)_\infty} = \frac{(1-tz)(1-tzq)(1-tzq^2)\cdots}{(1-uz)(1-uzq)(1-uzq^2)\cdots}$$

$$= \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{(tu^{-1};q)_r}{(q;q)_r} u^r z^r$$

where $(\alpha;q)_r = (1-\alpha)(1-\alpha q)(1-\alpha q^2)\cdots(1-\alpha q^{r-1})$

Proof of the infinite q-binomial theorem

Let $L(z;q,t,u) = \dfrac{(tz;q)_\infty}{(uz;q)_\infty}$. (which is the left-hand side)

This satisfies

$$L(z;q,t,u) = \frac{1-tz}{1-uz} L(qz;q,t,u)$$

(this is a q-difference equation).

In this way, we obtain a recurrence on the coefficients

$$L(z;q,t,u) = \sum_{r \in \mathbb{Z}_{\geq 0}} c_r(q,t,u)\, z^r$$

which is — analogue of Pascal's triangle.

$$c_r(q,t,u)q^r - t\, c_{r-1}(q,t,u)q^{r-1} = c_r(q,t,u) - u\, c_{r-1}(q,t,u)$$

Solving this recursion gives

$$c_r(q,t,u) = c_{r-1}(q,t,u) \cdot \frac{u - tq^{r-1}}{1-qq^{r-1}} = u^r \cdot \frac{(tu^{-1};q)_r}{(q;q)_r}$$