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Advanced Discrete Mathematics
Problem Session 16

e_r and characteristic polynomials

Recall that

$$\sum_{r=0}^{k} e_r(x_1,\dots,x_n) z^r = \prod_{i=1}^{n} (1+x_i z)$$

$$= (1+x_1 z)(1+x_2 z)\cdots(1+x_n z)$$

$$= \sum_{r=0}^{n} \left(\sum_{1\le i_1<i_2<\cdots<i_r\le n} x_{i_1} x_{i_2}\cdots x_{i_r}\right)\cdot z^r$$

and so

$$e_r(x_1,\dots,x_n) = \sum_{1\le i_1<i_2<\cdots<i_r\le n} x_{i_1} x_{i_2}\cdots x_{i_r}.$$

Setting $z=-1$, we get

$$\prod_{i=1}^{n}(1-x_i) = \sum_{r=0}^{n} e_r(x_1,\dots,x_n)(-1)^r.$$

Now, let $A\in M_{n\times n}(\mathbb{F})$ be diagonalisable. This means there exists $P\in GL_n(\mathbb{F})$ such that

$$PAP^{-1} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

with $\lambda_1,\dots,\lambda_n \in \mathbb{F}$.

The characteristic polynomial of A is $p_A(t) \in \mathbb{F}[t]$ such that

$$p_A(t) = \det(A - tI)$$

$$= \det(P(A-tI)P^{-1})$$

$$= \det(PAP^{-1} - tI)$$

$$= \det\left(\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} - \begin{pmatrix}t & & \\ & \ddots & \\ & & t\end{pmatrix}\right)$$

$$= \det\begin{pmatrix}\lambda_1 - t & & \\ & \ddots & \\ & & \lambda_n - t\end{pmatrix}$$

$$= (\lambda_1 - t)\cdots(\lambda_n - t)$$

$$= (-1)^n \prod_{i=1}^{n} (t - \lambda_i)$$

$$= (-1)^n t^n \prod_{i=1}^{n} (1 - \lambda_i t^{-1})$$

$$= (-1)^n t^n \sum_{r=0}^{n} e_r(\lambda_1 t^{-1}, \dots, \lambda_n t^{-1})(-1)^r$$

$$= (-1)^n t^n \sum_{r=0}^{n} e_r(\lambda_1, \dots, \lambda_n)\, t^{-r} (-1)^r$$

$$= \sum_{r=0}^{n} e_r(\lambda_1, \dots, \lambda_n)(-1)^{n-r} t^{n-r}$$

p_r and matrix powers

$$\mathrm{tr}(A^r) = \mathrm{tr}(PA^rP^{-1})$$

$$= \mathrm{tr}((PAP^{-1})^r)$$

$$= \mathrm{tr}\left(\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}^r\right)$$

$$= \mathrm{tr}\left(\begin{pmatrix} \lambda_1^r & & \\ & \ddots & \\ & & \lambda_n^r \end{pmatrix}\right)$$

$$= \lambda_1^r + \cdots + \lambda_n^r$$

$$= p_r(\lambda_1, \ldots, \lambda_n)$$

So, the question of

what is the relationship between $p_A(t)$ and $\mathrm{tr}(A^r)$

is equivalent to the question of

what is the relationship between e_r and p_r?

The answer is

the classical Newton identities $k e_k \sum_{i=1}^{k} (-1)^{i-1} p_i e_{k-i}$

There is also, in a similar vein

the Wronski identities $\sum_{i+j=k} (-1)^i e_i h_j = 0$.

which is actually found by taking the Euler characteristic of the Koszul/de Rham complex

$$0 \xrightarrow{d} \Lambda^r(\mathbb{F}^n) \otimes S^0(\mathbb{F}^n) \xrightarrow{d} \Lambda^{r-1}(\mathbb{F}^n) \otimes S^1(\mathbb{F}^n)$$

$$\xrightarrow{d} \cdots$$

$$\xrightarrow{d} \Lambda^1(\mathbb{F}^n) \otimes S^{r-1}(\mathbb{F}^n) \xrightarrow{d} \Lambda^0(\mathbb{F}^n) \otimes S^r(\mathbb{F}^n) \to 0.$$

<u>Total positivity</u>

This ONLY is for matrices $A \in M_{n\times n}(\mathbb{R})$ (other fields don't have a notion of 'positive').

We have the identity

$$e_r(\lambda_1, \ldots, \lambda_n) = tr(\Lambda^r(A))$$

Given $A \in M_{n\times n}(\mathbb{R})$ we get a linear transformation

$$A : \mathbb{R}^n \to \mathbb{R}^n$$

but we also get many extra linear transformations for free:

$$\Lambda^r(A) : \Lambda^r(\mathbb{R}^n) \longrightarrow \Lambda^r(\mathbb{R}^n)$$

where

$$\Lambda^r(\mathbb{R}^n) = \mathbb{R}\text{-span}\{b_{i_1} \wedge \cdots \wedge b_{i_r} \mid i_1, \ldots, i_r \in \{1, \ldots, n\},\ i_1 < \cdots < i_r\}$$

and

$$b_i = \underbrace{(0, \dots, 1, \dots, 0)}_{i^{\text{th}}\text{ position}}^T .$$

The $r \times r$ minors are the entries of $\Lambda^r(A)$ with respect to the above basis.

The game of total positivity is about answering the question of

Can you write a matrix with positive minors?