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Advanced Discrete Mathematics
Lecture 16

Recall

$$(z;q)_\infty = (1-z)(1-zq)(1-zq^2)\cdots$$

and we define $\tilde{g}_r$ by

$$\prod_{i=1}^{n} \frac{(tx_i z;q)_\infty}{(ux_i z;q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{g}_r z^r .$$

Cauchy-Macdonald Kernel

We wish to prove

$$\prod_{j=1}^{m}\prod_{i=1}^{n} \frac{(tx_i y_j;q)_\infty}{(ux_i y_j;q)_\infty} = \sum_{\substack{\lambda \\ l(\lambda)\leq n \\ l(\lambda)\leq m}} \tilde{g}_\lambda(x_1,\dots,x_n;q,t)\, m_\lambda(y_1,\dots,y_m;q,t)$$

$$= \sum_{\substack{\lambda \\ l(\lambda)\leq n \\ l(\lambda)\leq m}} \frac{1}{z_\lambda(q,t,u)} p_\lambda(x_1,\dots,x_n)\, p_\lambda(y_1,\dots,y_m)$$

where we remember that for $(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n$ with $\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n$ we have

$$m_\lambda = \sum_{\gamma\in S_n\lambda} x^\gamma, \qquad \text{where} \quad x^\gamma = x_1^{\gamma_1}\cdots x_n^{\gamma_n}.$$

The first equality comes from unraveling the definitions:

$$\prod_{j=1}^{m} \prod_{i=1}^{n} \frac{(tx_i y_j ; q)_\infty}{(ux_i y_j ; q)_\infty}$$

$$= \left(\prod_{i=1}^{n} \frac{(tx_i y_1 ; q)_\infty}{(ux_i y_1 ; q)_\infty} \right) \left(\prod_{i=1}^{n} \frac{(tx_i y_2 ; q)_\infty}{(ux_i y_2 ; q)_\infty} \right) \cdots \left(\prod_{i=1}^{n} \frac{(tx_i y_m ; q)_\infty}{(ux_i y_m ; q)_\infty} \right)$$

$$= \left(\sum_{r_1 \in \mathbb{Z}_{\geq 0}} \tilde{g}_{r_1}(x) y_1^{r_1} \right) \left(\sum_{r_2 \in \mathbb{Z}_{\geq 0}} \tilde{g}_{r_2}(x) y_2^{r_2} \right) \cdots \left(\sum_{r_m \in \mathbb{Z}_{\geq 0}} \tilde{g}_{r_m}(x) y_m^{r_m} \right)$$

$$= \sum_{r_1, r_2, \ldots, r_m \in \mathbb{Z}_{\geq 0}} \tilde{g}_{r_1}(x) \tilde{g}_{r_2}(x) \cdots \tilde{g}_{r_m}(x) y_1^{r_1} y_2^{r_2} \cdots y_m^{r_m}$$

$$= \sum_{\substack{\lambda \\ \ell(\lambda) \leq m}} \tilde{g}_\lambda(x) m_\lambda(y) .$$

For the second equality, recall the definitions

$$p_r(x_1, \ldots, x_n) = x_1^r + \cdots + x_n^r , \qquad p_\lambda(x) = p_{\lambda_1}(x) \cdots p_{\lambda_n}(x),$$

if $\lambda = (\lambda_1, \ldots, \lambda_n)$ and

$$z_\lambda(q, t, u) = z_\lambda \prod_{i=1}^{n} \left(\frac{1 - q^{\lambda_i}}{u^{\lambda_i} - t^{\lambda_i}} \right), \qquad z_\lambda = 1^{m_1} m_1! \cdot 2^{m_2} m_2! \, 3^{m_3} m_3! \cdots$$

if $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $m_j = \#$ parts of λ equal to j.

Further, recall that

$$\log(1-z) = \int \frac{-1}{1-z}\, dz$$

$$= \int -(1+z+z^2+z^3+\cdots)\, dz$$

$$= -z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \cdots$$

$$= \sum_{r\in\mathbb{Z}_{>0}} -\frac{1}{r} z^r .$$

This gives

$$\log\left(\prod_{i=1}^{n} \frac{(tx_iy;q)_\infty}{(ux_iy;q)_\infty}\right)$$

$$= \sum_{i=1}^{n} \log\left(\frac{(1-tx_iy)(1-tx_iyq)(1-tx_iyq^2)\cdots}{(1-ux_iy)(1-ux_iyq)(1-ux_iyq^2)\cdots}\right)$$

$$= \sum_{i=1}^{n} \sum_{l\in\mathbb{Z}_{\geq 0}} \log(1-tx_iyq^l) - \log(1-ux_iyq^l)$$

$$= \sum_{i=1}^{n} \sum_{l\in\mathbb{Z}_{\geq 0}} \left[\sum_{r\in\mathbb{Z}_{>0}} \frac{-1}{r} t^r x_i^r y^r q^{lr}\right] - \left[\sum_{r\in\mathbb{Z}_{>0}} -\frac{1}{r} u^r x_i^r y^r q^{lr}\right]$$

$$= \sum_{i=1}^{n} \sum_{l\in\mathbb{Z}_{\geq 0}} \sum_{r\in\mathbb{Z}_{>0}} \frac{1}{r} x_i^r y^r q^{lr} (u^r - t^r)$$

$$= \sum_{r \in \mathbb{Z}_{>0}} \frac{1}{r} y^r (u^r - t^r) \sum_{i=1}^{n} x_i^r \sum_{l \in \mathbb{Z}_{\geq 0}} (q^r)^l$$

$$= \sum_{r \in \mathbb{Z}_{>0}} \frac{1}{r} y^r (u^r - t^r) p_r(x) \cdot \frac{1}{1-q^r}$$

$$= \sum_{r \in \mathbb{Z}_{>0}} \left(\frac{u^r - t^r}{1-q^r} \right) \frac{p_r(x)}{r} y^r$$

So

$$\prod_{i=1}^{n} \frac{(tx_i y; q)_\infty}{(ux_i y; q)_\infty} = \exp\left(\log\left(\prod_{i=1}^{n} \frac{(tx_i y; q)_\infty}{(ux_i y; q)_\infty} \right)\right)$$

$$= \exp\left(\sum_{r \in \mathbb{Z}_{>0}} \left(\frac{u^r - t^r}{1-q^r} \right) \frac{p_r(x)}{r} y^r \right)$$

$$= \prod_{r \in \mathbb{Z}_{>0}} \exp\left(\left(\frac{u^r - t^r}{1-q^r} \right) \frac{p_r(x)}{r} y^r \right)$$

$$= \prod_{r \in \mathbb{Z}_{>0}} \left(1 + \frac{1}{1!} \left(\frac{u^r - t^r}{1-q^r} \right)^1 \frac{p_r(x)^1}{r^1} y^{r \cdot 1} + \frac{1}{2!} \left(\frac{u^r - t^r}{1-q^r} \right)^2 \frac{p_r(x)^2}{r^2} y^{r \cdot 2} + \cdots \right)$$

(we expand by choosing how many $p_1(x)$s we want (i.e how many 1s in our partition), how many $p_2(x)$s we want (i.e. how many 2s in our partition) and so on.)

$$= \sum_{m_1, m_2 \in \mathbb{Z}_{\geq 0}} \left[\frac{1}{m_1!} \left(\frac{u^1 - t^1}{1-q^1} \right)^{m_1} \frac{p_1(x)^{m_1}}{1^{m_1}} \cdot y^{1 \cdot m_1} \right] \left[\frac{1}{m_2!} \left(\frac{u^2 - t^2}{1-q^2} \right)^{m_2} \frac{p_2(x)^{m_2}}{2^{m_2}} \cdot y^{2 \cdot m_2} \right]$$

$$\cdots$$

$$= \sum_{m_1, m_2 \ldots \in \mathbb{Z}_{\geq 0}} \left[\frac{1}{1^{m_1} \cdot m_1! \cdot 2^{m_2} \cdot m_2! \cdots}\right]\left[\left(\frac{u^1-t^1}{1-q^1}\right)^{m_1}\left(\frac{u^2-t^2}{1-q^2}\right)^{m_2} \cdots\right]\left[p_1(x)^{m_1} p_2(x)^{m_2} \cdots\right]$$

$$\left[y^{1 \cdot m_1 + 2m_2 + \cdots}\right]$$

$$= \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{\substack{m_1, m_2 \ldots \in \mathbb{Z}_{\geq 0} \\ m_1 + 2m_2 + 3m_3 + \cdots = n}} \left[\frac{1}{1^{m_1} m_1! 2^{m_2} m_2! \cdots}\right]\left[\left(\frac{u^1-t^1}{1-q^1}\right)^{m_1}\left(\frac{u^2-t^2}{1-q^2}\right)^{m_2} \cdots\right]\left[p_1(x)^{m_1} p_2(x)^{m_2} \cdots\right]$$

$$\cdot \; y^n$$

$$= \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{\substack{\lambda \\ |\lambda| = n}} \frac{1}{z_\lambda} \prod_{k \in \mathbb{Z}_{\geq 1}} \left(\frac{u^k - t^k}{1-q^k}\right)^{\lambda_k} p_\lambda(x) \cdot y^n .$$

$$= \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{\substack{\lambda \\ |\lambda| = n}} \frac{1}{z_\lambda} \prod_{i=1}^{\ell(\lambda)} \left(\frac{u^{\lambda_i} - t^{\lambda_i}}{1-q^{\lambda_i}}\right) p_\lambda(x) \cdot y^n$$

$$= \sum_{n \in \mathbb{Z}_{\geq 0}} \sum_{\substack{\lambda \\ |\lambda| = n}} \frac{1}{z_\lambda(q,t,u)} \cdot p_\lambda(x) \cdot y^n .$$

This completes the bulk of the proof. $\square$