Advanced Discrete Mathematics
Problem Solving 15

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29th August 2025

How do you prove Bruhat decomposition?

Let

$\mathbb{F}$ be a field
$G = GL_n(\mathbb{F})$ the general linear group
$B \leq GL_n(\mathbb{F})$ the subgroup of upper triangular matrices
$S_n \leq GL_n(\mathbb{F})$ the subgroup of permutation matrices

Then

$$G = \bigsqcup_{w \in S_n} BwB.$$

We should try to do this for $n=2$ and $n=3$.

For $n=2$: $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $s_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $S_2 = \{1, s_1\}$.

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \;\middle|\; \begin{array}{c} a, c \in \mathbb{F}^\times \\ b \in \mathbb{F} \end{array} \right\}.$$

So the claim is

$$GL_2(\mathbb{F}) = B \cdot 1 \cdot B \sqcup Bs_1B$$

$$= B \sqcup Bs_1B.$$

Define, for $c \in \mathbb{F}$

$$y_1(c) = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} \qquad x_{12}(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \qquad x_{21}(c) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

Since

$$\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then we also have

$$y_1(c)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$$

then

$$x_{12}(a)\, x_{12}(b) = x_{12}(a+b).$$

Since

$$\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$x_{12}(c)\, s_1 = y_1(c).$$

Since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & c \end{pmatrix}$$

then

$$\underline{s_1 x_{12}(c) = x_{21}(c)\, s_1}.$$

In fact, this is a general phenomenon: if $w \in S_n$ and $i, j \in \{1, \dots, n\}$ then

$$\underline{w\, x_{ij}(c) = x_{w(i)w(j)}(c)\, w}$$

(for a suitable choice of definition for $x_{ij}(c)$).

Steps to do:

(a) $\bigcup_{w \in S_n} BwB \subseteq G$ (basic group theory)

(b) $G \subseteq \bigcup_{w \in S_n} BwB$ (the hard bit!)

(c) If $w_1, w_2 \in S_n$ and $w_1 \neq w_2$ then $(Bw_1B) \cap (Bw_2B) = \emptyset$

Let us now consider the $\underline{n=2}$ case for

$$g = \begin{pmatrix} 5 & 7 \\ 6 & -10 \end{pmatrix}$$

Our goal is to find $b_1, b_2 \in B$ and $w \in S_2$ such that $g = b_1 w b_2$.

Let us perform row reduction.
Our goal is to remove the $(2,1)$ entry:

$$\begin{pmatrix} 5 & 7 \\ 6 & -10 \end{pmatrix} \rightsquigarrow \begin{pmatrix} ? & ? \\ 0 & ? \end{pmatrix}$$

We do this by multiplying on the left by $y_1(c_1)^{-1}$:

$$\begin{pmatrix} 0 & 1 \\ 1 & c_1 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 6 & -10 \end{pmatrix} = \begin{pmatrix} 6 & -10 \\ 5+6c_1 & 7-10c_1 \end{pmatrix}$$

We have

$$5+6c_1 = 0 \Rightarrow 6c_1 = -5 \Rightarrow c_1 = -5/6$$

So

$$y_1(-5/6)^{-1} g = \begin{pmatrix} 6 & -10 \\ 0 & 46/3 \end{pmatrix}$$

Define

$$b_2 = \begin{pmatrix} 6 & -10 \\ 0 & 46/3 \end{pmatrix}$$

So

$$y_1(-5/6)^{-1} g = b_2$$

$$\Leftrightarrow g = y_1(-5/6)\, b_2$$

$$= x_{12}(-5/6)\, s_1\, b_2$$

Defining $b_1 = x_{12}(-5/6)$ gives

$$g = b_1 s_1 b_2 \in \mathcal{B} s_1 \mathcal{B}$$

as required.

<u>For $n=3$</u>

$$S_3 = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}, \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

$$= \{ 1, s_1, s_2, s_1 s_2 s_1, s_1 s_2, s_2 s_1 \}$$

The claim is now

$$G = (\mathcal{B} 1 \mathcal{B}) \sqcup (\mathcal{B} s_1 \mathcal{B}) \sqcup (\mathcal{B} s_2 \mathcal{B}) \sqcup (\mathcal{B} s_1 s_2 s_1 \mathcal{B}) \sqcup (\mathcal{B} s_1 s_2 \mathcal{B}) \sqcup (\mathcal{B} s_2 s_1 \mathcal{B})$$

where

$$\mathcal{B} 1 \mathcal{B} = \mathcal{B} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \middle| \begin{array}{l} a, d, f \in \mathbb{F}^\times \\ b, c, e \in \mathbb{F} \end{array} \right\}$$

$$\mathcal{B} s_1 \mathcal{B} = \left\{ \begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| c_1 \in \mathbb{F} \right\}$$

notice that we have 0s to the right and below 1s!

$$Bs_2B = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \middle| \; c_1 \in \mathbb{F} \right\}$$

$$Bs_1s_2s_1B = \left\{ \begin{pmatrix} c_1 & c_2 & 1 \\ c_3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \middle| \; c_1, c_2, c_3 \in \mathbb{F} \right\}$$

$$Bs_1s_2B = \left\{ \begin{pmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \middle| \; c_1, c_2 \in \mathbb{F} \right\}$$

$$Bs_2s_1B = \left\{ \begin{pmatrix} c_1 & c_2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \middle| \; c_1, c_2 \in \mathbb{F} \right\}$$

Let us consider this case for

$$g = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$$

Our goal will be:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} \rightsquigarrow \begin{pmatrix} ? & ? & ? \\ ? & ? & ? \\ 0 & ? & ? \end{pmatrix} \rightsquigarrow \begin{pmatrix} ? & ? & ? \\ 0 & ? & ? \\ 0 & ? & ? \end{pmatrix} \rightsquigarrow \begin{pmatrix} ? & ? & ? \\ 0 & ? & ? \\ 0 & 0 & ? \end{pmatrix}$$

(the flow is to start in the bottom left corner and travel upwards)

Define $\quad$ the subscript indicates which row you are removing!

$$y_1(c) = \begin{pmatrix} c & 1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} \quad \text{so} \quad y_1(c)^{-1} = \begin{pmatrix} 0 & 1 & \\ 1 & -c & \\ & & 1 \end{pmatrix}$$

and

$$y_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{so} \quad y_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c \end{pmatrix}$$

and

$x_{ij} = 1 + cE_{ij}$ for $c \in F$ and $i, j \in \{1, \ldots, 3\}$ and $i \neq j$.

We have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c_1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 10 \\ 4-7c_1 & 5-8c_1 & 6-10c_1 \end{pmatrix}$$

This means we want

$$4 - 7c_1 = 0 \Rightarrow 7c_1 = 4 \Rightarrow c_1 = 4/7$$

So

$$y_2(4/7)^{-1} g = \begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 10 \\ 0 & 3/7 & 2/7 \end{pmatrix}$$

We have

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 7 & 8 & 10 \\ 0 & 3/7 & 2/7 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 10 \\ 1-7c_2 & 2-8c_2 & 3-10c_2 \\ 0 & 3/7 & 2/7 \end{pmatrix}.$$

This means we want

$$1-7c_2 = 0 \Rightarrow 7c_2 = 1 \Rightarrow c_2 = 1/7.$$

So

$$y_1(1/7)^{-1} y_2(4/7)^{-1} g = \begin{pmatrix} 7 & 8 & 10 \\ 0 & 6/7 & 11/7 \\ 0 & 3/7 & 2/7 \end{pmatrix}.$$

We have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c_3 \end{pmatrix}\begin{pmatrix} 7 & 8 & 10 \\ 0 & 6/7 & 11/7 \\ 0 & 3/7 & 2/7 \end{pmatrix} = \begin{pmatrix} 7 & 8 & 10 \\ 0 & 3/7 & 2/7 \\ 0 & 6/7 - 3/7 c_3 & 11/7 - 2/7 c_3 \end{pmatrix}$$

This means we want

$$6/7 - 3/7 c_3 = 0 \Rightarrow \tfrac{3}{7} c_3 = 6/7 \Rightarrow 3c_3 = 6 \Rightarrow c_3 = 2$$

So

$$y_2(2)^{-1} y_1(1/7)^{-1} y_2(4/7)^{-1} g = \begin{pmatrix} 7 & 8 & 10 \\ 0 & 3/7 & 2/7 \\ 0 & 0 & 1 \end{pmatrix}$$

Define

$$b_2 = \begin{pmatrix} 7 & 8 & 10 \\ 0 & 3/7 & 2/7 \\ 0 & 0 & 1 \end{pmatrix}$$

So

$$g = y_2(4/7)\,y_1(1/7)\,y_2(2)\,b_2$$

Like the $n=2$ case, we have the identities

$$\underline{y_k(c) = x_{k\,k+1}(c)\,s_k}, \qquad k \in \{1, \dots, n-1\}.$$

So

$$\begin{aligned} g &= y_2(4/7)\,y_1(1/7)\,y_2(2)\,b_2 \\ &= x_{23}(4/7)\,s_2\,x_{12}(1/7)\,s_1\,x_{23}(2)\,s_2\,b_2 \\ &= x_{23}(4/7)\,x_{13}(1/7)\,s_2 s_1\,x_{23}(2)\,s_2\,b_2 \\ &= x_{23}(4/7)\,x_{13}(1/7)\,s_2\,x_{13}(2)\,s_1 s_2\,b_2 \\ &= x_{23}(4/7)\,x_{13}(1/7)\,x_{12}(2)\,s_2 s_1 s_2\,b_2 \\ &= x_{23}(4/7)\,x_{13}(1/7)\,x_{12}(2)\,s_1 s_2 s_1\,b_2 \end{aligned}$$

Define $b_1 = x_{23}(4/7)\,x_{13}(1/7)\,x_{12}(2)$.

$w = s_1 s_2 s_1$

So

$$b_1 = \begin{pmatrix} 1 & 2 & 1/7 \\ 0 & 1 & 4/7 \\ 0 & 0 & 1 \end{pmatrix}$$

We have

$$b_1 w b_2 =$$

$$\begin{pmatrix} 1 & 2 & -1/7 \\ 0 & 1 & 4/7 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 7 & 8 & 10 \\ 0 & 3/7 & 2/7 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/7 & 2 & 1 \\ 4/7 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 7 & 8 & 10 \\ 0 & 3/7 & 2/7 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$$

$$= g.$$

<u>Does Bruhat decomposition give topological information?</u>

If $\mathbb{F}$ has a topology such that $\overline{\mathbb{F} - \{0\}} = \mathbb{F}$ then

$$\overline{B_{s,B}} = B_{s,B} \cup B$$

in $GL_2(\mathbb{F})$.

More generally, the Bruhat order on S_n is given by

$$v \le w \quad \text{if} \quad BvB \subseteq \overline{BwB} = \bigsqcup_{v \le w} BvB.$$

For $GL_3(\mathbb{F})$ we have