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Advanced Discrete Mathematics
Lecture 15

Monomial Symmetric Functions

For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$, the monomial symmetric functions can be defined in two ways:

(1) $m_\lambda = \frac{1}{v_\lambda} e x^\lambda$ where $e = \sum_{w \in S_n} w$ and $v_\lambda \in \mathbb{Z}$ is chosen so that the coefficient of x^λ in m_λ is 1.

(2) $m_\lambda = \sum_{\gamma \in S_n \lambda} x^\gamma$ (no worries about normalization!)

Recall

S_n acts on $\mathbb{Z}^n$ by permuting the entries

$$w(\gamma_1, \dots, \gamma_n) = (\gamma_{w(1)}, \dots, \gamma_{w(n)})$$

In this context,

$$S_n \lambda = \text{orbit of } \lambda \text{ under } S_n = \{ w\lambda \mid w \in S_n \}.$$

So this forces the coefficient of x^λ in $\sum_{\gamma \in S_n \lambda} x^\gamma$ to be 1.

Notice by orbit-stabiliser: $\text{Stab}(\lambda) = \{ w \in S_n \mid w\lambda = \lambda \}$ and $v_\lambda = \text{Card}(\text{Stab}(\lambda))$.

For example, take $\lambda = (4,0,0)$ and $n=3$.

Then, $S_n\lambda = \{(4,0,0), (0,4,0), (0,0,4)\}$.
We have $v_\lambda = \mathrm{Card}(\mathrm{Stab}((4,0,0)))$
$$= \mathrm{Card}(\{\equiv \overline{\times}\})$$
$$= 2.$$

This coincides with last lecture's result that

$$m_{(4,0,0)} = x_1^4 + x_2^4 + x_3^4.$$

Remarkable fact: The set

$$\{m_\lambda \mid \lambda \in \mathbb{Z}^n \text{ with } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n\}$$

is a basis of $\mathbb{C}[x_1,\dots,x_n]^{S_n}$.

q-binomial theorem

Recall that $\tilde{g}_r(x;q,t,u)$ is given by

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{g}_r z^r = \prod_{i=1}^{n} \frac{(tx_i z;q)_\infty}{(ux_i z;q)_\infty}$$

$$= \prod_{i=1}^{n} \frac{(1-tx_i z)(1-tx_i zq)(1-tx_i zq^2)\cdots}{(1-ux_i z)(1-ux_i zq)(1-ux_i zq^2)\cdots}$$

The q-binomial theorem is the following remarkable statement

$$\frac{(tx_i z;q)_\infty}{(x_i z;q)_\infty} = \sum_{r\in\mathbb{Z}_{\geq 0}} \frac{(t;q)_r}{(q;q)_r} x_i^r z^r \quad \text{where}$$

$$(t;q)_r = (1-t)(1-tq)\cdots(1-tq^{r-1})$$

and

$$(q;q)_r = (1-q)(1-q^2)\cdots(1-q^r).$$

The coefficients $\frac{(t;q)_r}{(q;q)_r}$ are analogous to the binomial coefficients

and so one might expect a version of Pascal's triangle.

Applying the q-binomial theorem

We have

$$\prod_{i=1}^{n} \frac{(tx_i z;q)_\infty}{(x_i z;q)_\infty}$$

$$= \frac{(tx_1 z;q)_\infty}{(x_1 z;q)_\infty} \cdots \frac{(tx_n z;q)_\infty}{(x_n z;q)_\infty}$$

$$= \sum_{\mu_1,\dots,\mu_n \in \mathbb{Z}_{\geq 0}} \frac{(t;q)_{\mu_1}}{(q;q)_{\mu_1}} \cdots \frac{(t;q)_{\mu_n}}{(q;q)_{\mu_n}} x_1^{\mu_1}\cdots x_n^{\mu_n} z^{\mu_1+\cdots+\mu_n}$$

$$= \sum_{r\in\mathbb{Z}_{\geq 0}} \sum_{\substack{(\mu_1,\dots,\mu_n)\in\mathbb{Z}_{\geq 0}^n \\ \mu_1+\cdots+\mu_n=r}} \frac{(t;q)_{\mu_1}}{(q;q)_{\mu_1}} \cdots \frac{(t;q)_{\mu_n}}{(q;q)_{\mu_n}} x^\mu z^r$$

$$= \sum_{r\in\mathbb{Z}_{\geq 0}} \sum_{\substack{(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}_{\geq 0}^n \\ \lambda_1+\cdots+\lambda_n=r \\ \lambda_1\geq\cdots\geq\lambda_n}} \frac{(t;q)_{\lambda_1}}{(q;q)_{\lambda_1}} \cdots \frac{(t;q)_{\lambda_n}}{(q;q)_{\lambda_n}} m_\lambda z^r$$

So one concludes that

$$g_r = \sum_{\substack{\lambda \text{ partitions} \\ |\lambda| = r,\ \ell(\lambda) \le n}} \frac{(t;q)_{\lambda_1}}{(q;q)_{\lambda_1}} \cdots \frac{(t;q)_{\lambda_n}}{(q;q)_{\lambda_n}} m_\lambda$$

(this is an explicit formula for the one-row Macdonald polynomials!)

Power sums and the Cauchy-Macdonald kernel

The **power sums** are

$$p_0 = 1 \qquad \text{and} \qquad p_r = x_1^r + \cdots + x_n^r$$

for $r \in \mathbb{Z}_{>0}$.

If $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\ge 0}^n$ with $\lambda_1 \ge \cdots \ge \lambda_n$ then

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_n}$$

and

$$\tilde{g}_\lambda = \tilde{g}_{\lambda_1} \cdots \tilde{g}_{\lambda_n}$$

Theorem (Cauchy-Macdonald Identity)

$$\prod_{j=1}^{n} \prod_{i=1}^{n} \frac{(t x_i y_j; q)_\infty}{(u x_i y_j; q)_\infty} = \sum_{\substack{\lambda \text{ partitions} \\ \ell(\lambda) \le n}} \tilde{g}_\lambda(x_1, \ldots, x_n; q, t, u)\, m_\lambda(y_1, \ldots, y_n)$$

$$= \sum_{\substack{\lambda \text{ partition} \\ \ell(\lambda) \le n}} \frac{1}{z_\lambda(q,t,u)} p_\lambda(x_1, \ldots, x_n)\, p_\lambda(x_1, \ldots, x_n).$$

where $z_\lambda = (1^{m_1} m_1!)(2^{m_2} m_2!)\cdots(k^{m_k} k!)$

and $m_k = \mathrm{Card}(\{i \mid \lambda_i = k\})$

$=$ # of parts in $(\lambda_1,\dots,\lambda_n)$ that are equal to k.

For example, if $\lambda = (5,5,3,3,3,1)$ then

$m_1 = 1,\ m_2 = 0,\ m_3 = 3,\ m_4 = 0,\ m_5 = 2,\ m_k = 0,\ k \geq 6.$

So, $z_{(5,5,3,3,3,1)} = 1^1 \cdot 1! \cdot 3^3 \cdot 3! \cdot 5^2 \cdot 2!$

The Macdonald inner product is given by one of the following equivalent definitions:

(a) $\langle \tilde{g}_\lambda, m_\mu \rangle_M = \delta_{\lambda\mu}$

(b) $\langle p_\lambda, p_\mu \rangle_M = \delta_{\lambda\mu} \cdot z_\lambda(q,t,u)$

(c) $\langle P_\lambda, P_\mu \rangle = \delta_{\lambda\mu}$

Similarly, we have the Hall inner product:

(a) $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}$

(b) $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \cdot z_\lambda$

(c) $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$