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Advanced Discrete Mathematics
Lecture 14

$\tilde{g}_r = g_r(x;q,t,u) = \tilde{g}_r(x_1,\dots,x_n;q,t,u)$ are generated by

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{g}_r z^r = \prod_{i=1}^{n} \frac{(tx_i z;q)_\infty}{(ux_i z;q)_\infty}$$

$$= \prod_{i=1}^{n} \frac{(1-tx_i z)(1-tx_i zq)(1-tx_i zq^2)\cdots}{(1-ux_i z)(1-ux_i zq)(1-ux_i zq^2)\cdots}$$

We have the degenerations:

$$\tilde{g}_r(x;t,u) = \tilde{g}_r(x;0,t,u)$$

$$h_r(x) = \tilde{g}_r(x;0,0,1)$$

$$q_r(x,t) = \tilde{g}_r(x;0,t,1)$$

$$e_r(x) = \tilde{g}_r(x;0,-1,0)$$

We aim to find an explicit formula for $\tilde{g}_r$, but let us find $\tilde{g}_r$ first.

First, $$\frac{1}{1-ux_i z} = 1 + ux_i z + u^2 x_i^2 z^2 + \cdots$$

So, $$\frac{1-tx_i z}{1-ux_i z} = (1-tx_i z)(1+ux_i z + u^2 x_i^2 z^2 + u^3 x_i^3 z^3 + \cdots)$$

$$= 1 + ux_i z + u^2 x_i^2 z^2 + u^3 x_i^3 z^3 + \cdots$$
$$\quad - tx_i z - tux_i^2 z^2 - tu^2 x_i^3 z^3 - tu^3 x_i^4 z^4 - \cdots$$

$$= 1 + u\left(x_i z + u x_i^2 z^2 + u^2 x_i^3 z^3 + \cdots\right)$$
$$- t\left(x_i z + u x_i^2 z^2 + u^2 x_i^3 z^3 + \cdots\right)$$

$$= 1 + (u-t)\left(x_i z + u x_i^2 z^2 + u^2 x_i^3 z^3 + \cdots\right)$$

$$= 1 + (u-t) x_i z \left(1 + u x_i z + u^2 x_i^2 z^2 + \cdots\right)$$

Then,

$$\sum_{r \in \mathbb{Z}_{\geq 0}} \tilde{q}_r z^r = \prod_{i=1}^{n} \frac{1 - t x_i z}{1 - u x_i z}$$

$$= \left(\frac{1 - t x_1 z}{1 - u x_1 z}\right) \cdots \left(\frac{1 - t x_n z}{1 - u x_n z}\right)$$

$$= \left[1 + (u-t) x_1 z \left(1 + u x_1 z + u^2 x_1^2 z^2 + \cdots\right)\right]$$
$$\cdots \left[1 + (u-t) x_n z \left(1 + u x_n z + u^2 x_n^2 z^2 + \cdots\right)\right]$$

(and, after some MANIPULATORICS...)

$$\tilde{q}_r = \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} (u-t)^{1 + \#\{j \mid i_j < i_{j+1}\}} u^{\#\{j \mid i_j = i_{j+1}\}} x_{i_1} \cdots x_{i_r}$$

Then,

$$ev_{t=u}\left(\frac{1}{u-t} \tilde{q}_r\right) = ev_{t=u}\left(\frac{1}{\cancel{u-t}}\left[\sum_{1 \leq i_1 \leq \cdots \leq i_r \leq n} (u-t)^{\cancel{1+}\#\{j \mid i_j < i_{j+1}\}}\right.\right.$$
$$\left.\left. u^{\#\{j \mid i_j = i_{j+1}\}} x_{i_1} \cdots x_{i_r}\right]\right)$$

$$= \sum_{i=1}^{n} u^{r-1} x_i^r = u^{r-1} \sum_{i=1}^{n} x_i^r = u^{r-1} p_r.$$

A miracle! You can get the power sum symmetric functions from the gs!

<u>The monomial symmetric functions m_λ</u>

Recall that S_n acts on $\mathbb{C}[x_1, \dots, x_n]$ by

$$(wf)(x_1, \dots, x_n) = f(x_{w(1)}, \dots, x_{w(n)})$$

and

$$\mathbb{C}[x_1, \dots, x_n]^{S_n} = \{ f \in \mathbb{C}[x_1, \dots, x_n] \mid \text{if } w \in S_n \text{ then } wf = f \}$$

Take $f = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$ where $\mu = (\mu_1, \dots, \mu_n)$.

(for example if $n=3$ and $\mu = (5,0,7)$ then $x^\mu = x^{(5,0,7)} = x_1^5 x_3^7$)

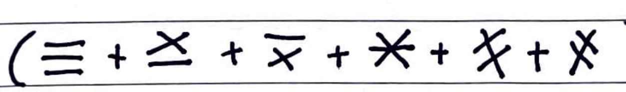
Define the monomial symmetric functions $\underline{m_\lambda}$ such that

$$m_\lambda = \frac{1}{v_\lambda} e x^\lambda ,$$

where

$$e = \sum_{w \in S_n} w \quad \text{and} \quad v_\lambda \in \mathbb{Z} \text{ is chosen so the coefficient of } x^\lambda \text{ in } m_\lambda \text{ is } 1.$$

<u>For example,</u> suppose $\mu = (5,0,7)$. Then

$ex^\mu = \Big($  $\Big) x_1^5 x_3^7$

$$= x_1^3 x_3^7 + x_2^3 x_3^7 + x_1^3 x_2^7 + x_3^3 x_1^7 + x_2^3 x_1^7 + x_3^3 x_2^7$$

In this case, we have $v_\mu = 1$ and so

$$m_\mu = m_{(5,0,7)} = x_1^3 x_3^7 + x_2^3 x_3^7 + x_1^3 x_2^7 + x_3^3 x_1^7 + x_2^3 x_1^7 + x_3^3 x_2^7.$$

Another example. Suppose $\mu = (4,0,0)$ and $n=3$. Then

$$ex^\mu = (\quad + \quad + \quad + \quad + \quad + \quad) x_1^4$$

$$= x_1^4 + x_2^4 + x_1^4 + x_3^4 + x_2^4 + x_3^4$$

$$= 2(x_1^4 + x_2^4 + x_3^4).$$

In this case, $v_\mu = 2$ and so

$$m_\mu = m_{(4,0,0)} = x_1^4 + x_2^4 + x_3^4.$$