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Advanced Discrete Mathematics
Lecture 13

Symmetric Functions

Let

$$\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = \mathbb{C}\text{-span}\{x_1^{\gamma_1} x_2^{\gamma_2} \cdots x_n^{\gamma_n} \mid \gamma_1, \dots, \gamma_n \in \mathbb{Z}\}$$

be the ring of Laurent polynomials in variables $x_1, \dots, x_n$.

For example, $p = x_1^2 x_2 + 7x_1 x_3^4 \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$.

The symmetric group acts on $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ by permuting the variables i.e.

$$S_n \times \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \longrightarrow \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

where

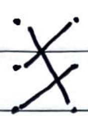
$$(wp)(x_1, \dots, x_n) = p(x_{w(1)}, \dots, x_{w(n)}).$$

One remarkable property of this action is that it induces an algebra homomorphism. Suppose $w \in S_n$ is fixed. Then,

$$w : \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}] \longrightarrow \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

satisfies

$$w(p_1 p_2) = w(p_1) w(p_2).$$

For example, if $w =$  and $p = x_1^2 x_2 + 7x_1 x_3^4$ then

$$wp = x_2^2 x_3 + 7 x_2 x_1^4.$$

The ring of symmetric functions is

$$\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{S_n} = \{ p \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \mid \text{if } w \in S_n \text{ then } wp = p \}.$$

An example of a family of symmetric functions is the power sum symmetric functions:

$$p_r = x_1^r + \cdots + x_n^r, \quad \text{for } r \in \mathbb{Z}.$$

In $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$,

$$p_1 = x_1 + x_2 + x_3$$

$$p_2 = x_1^2 + x_2^2 + x_3^2$$

$$p_3 = x_1^3 + x_2^3 + x_3^3, \quad \text{and so on} \ldots$$

Other examples include:

(a) The complete (also known as homogeneous) symmetric functions h_r

(b) The elementary symmetric functions e_r

(c) The monomial symmetric functions m_λ

We will see, however, that h_r, e_r are instances of

g_r, the one-row Macdonald polynomials

Define

$$(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\cdots$$

Let $q, t \in \mathbb{C}$.
Define

$$g_r = g_r(x;q,t) = g_r(x_1,\dots,x_n;q,t) \in \mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$$

by the following generating function

$$\sum_{r\in\mathbb{Z}_{\geq 0}} g_r z^r = \prod_{i=1}^{n} \frac{(tx_i z;q)_\infty}{(x_i z;q)_\infty}$$

$$= \prod_{i=1}^{n} \frac{(1-tx_i z)\cdot(1-tx_i zq)\cdot(1-tx_i zq^2)\cdots}{(1-x_i z)\cdot(1-x_i zq)\cdot(1-x_i zq^2)\cdots}$$

Define q_r, e_r, h_r by

$$\sum_{r\in\mathbb{Z}_{\geq 0}} q_r z^r = \prod_{i=1}^{n} \frac{1-tx_i z}{1-x_i z}, \qquad \sum_{r\in\mathbb{Z}_{\geq 0}} h_r z^r = \prod_{i=1}^{n} \frac{1}{1-x_i z}$$

$$\sum_{r\in\mathbb{Z}_{\geq 0}} e_r z^r = \prod_{i=1}^{n} (1+x_i z).$$

For example, if $n=3$ then

$$\sum_{r\in\mathbb{Z}_{\geq 0}} e_r z^r = \prod_{i=1}^{3}(1+x_i z)$$

$$= (1+x_1 z)(1+x_2 z)(1+x_3 z)$$

$$= 1 + (x_1+x_2+x_3)z + (x_1x_2 + x_1x_3 + x_2x_3)z^2$$

$$+ (x_1x_2x_3)z^3$$

So, in $\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ we have

$$e_1 = x_1 + x_2 + x_3$$

$$e_2 = x_1x_2 + x_1x_3 + x_2x_3$$

$$e_3 = x_1x_2x_3$$

$$e_r = 0 \quad \text{for } r \in \mathbb{Z}_{\geq 4}.$$

"Extensions"

Define $\tilde{g}_r = \tilde{g}_r(x;q,t,u)$ and $\hat{g}_r = \hat{g}_r(x;q,t,u)$ given by generating functions

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \tilde{g}_r z^r = \prod_{i=1}^{n} \frac{(tx_i z;q)_\infty}{(ux_i z;q)_\infty} \quad \text{and}$$

$$\sum_{r\in\mathbb{Z}_{\geq 0}} \hat{g}_r z^r = \prod_{i=1}^{n} \frac{1-tx_i z}{1-ux_i z}.$$

Then,

$$\tilde{q}_r(x;t,u) = \tilde{g}_r(x;0,t,u) \qquad q_r(x;t) = \tilde{g}_r(x;0,t,1)$$

$$h_r(x) = \tilde{g}_r(x;0,0,1) \qquad e_r(x) = \tilde{g}_r(x;0,-1,0).$$

Notice that

$$g_r(x;q,t) = \tilde{g}_r(x;q,t,1) \quad \text{and}$$

$$\tilde{g}_r(x_1,\dots,x_n;q,t,u) = u^r\,\tilde{g}_r(u^{-1}x_1,\dots,u^{-1}x_n;q,t,u)$$

$$= u^r\,g_r(x_1,\dots,x_n;q,tu^{-1}).$$

In this way, any identity involving $\tilde{g}_r$ immediately gives an identity involving g_r and vice versa.

Because of the connections to representation theory of S_n and $GL_n(\mathbb{C})$, lots of focus is on the Schur functions s_λ.

Macdonald generalised Schur functions to Macdonald polynomials.

$P_\lambda(x;q,t)$ Macdonald Polynomials

$P_\lambda(x;0,t)$ Hall-Littlewood Polynomials

$s_\lambda(x) = P_\lambda(x;0,0)$ Schur functions

In a similar way, the g_r construction gives

$g_r = P_{(r)}(x;q,t)$ One-row Macdonald Polynomials

$q_r = P_{(r)}(x;0,t)$ One-row Hall-Littlewood Polynomials

$h_r = P_{(r)}(x;0,0)$ One-row Schur functions.

Turning our attention to columns, we find the following miracle:

$P_{(1^r)}(x;q,t)$ One-column Macdonald polynomial

$= P_{(1^r)}(x;0,t)$ One-column Hall-Littlewood polynomial

$= P_{(1^r)}(x;0,0)$ One-column Schur functions

$= e_r$ elementary symmetric functions.

They all turn out to be equivalent!