Gypsy Akhyar
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Advanced Discrete Mathematics
Lecture 11

The Hecke Algebra

Let $q \in \mathbb{C}$ and $n \in \mathbb{Z}_{>0}$.
The **Hecke algebra** H is the $\mathbb{C}$-algebra generated by

$$T_1, \dots, T_{n-1}$$

with relations

(a) $T_j^2 = (q-1)T_j + q$ (Hecke relation)

(b) $T_j T_k = T_k T_j$, if $k \notin \{j-1, j+1\}$

(c) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ (braid relation or Artin relation or quantum Yang-Baxter)

Coxeter relations

for $j, k \in \{1, \dots, n-1\}$ and $i \in \{1, \dots, n-2\}$.

Presentations have wonderful simplicity (an abstract game).
Representation theory would like to make this more concrete.
Can we find $d \times d$ matrices with entries in $\mathbb{C}$

$$T_1, \dots, T_{n-1}$$

that satisfy (a), (b) and (c)?

Theorem Let $\mathbb{F}_q$ be a field with q elements.
Let $G(\mathbb{F}_q^n)$ be the $\mathbb{F}_q$-subspace lattice of $\mathbb{F}_q^n$ (partially ordered by inclusion).
Define

$$G/B = \{\text{maximal chains in } G(\mathbb{F}_q^n)\}$$
$$\mathbb{C}[G/B] = \mathbb{C}\text{-span}(G/B)$$

For $i \in \{1, \ldots, n-1\}$ define a linear transformation

$$T_i : \mathbb{C}[G/B] \longrightarrow \mathbb{C}[G/B]$$

by

$$T_i(0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = \sum_{\substack{V_{i-1} \subseteq W \subseteq V_{i+1} \\ W \neq V_i}} (0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{i-1} \subsetneq W \subsetneq V_{i+1} \subsetneq \cdots \subsetneq V_n).$$

Then, the linear transformations satisfy all the relations in the definition of the Hecke algebra.

Remark $GL_n(\mathbb{F}_q)$ acts on $G(\mathbb{F}_q^n)$ by

$$\begin{aligned} GL_n(\mathbb{F}_q) \times G(\mathbb{F}_q^n) &\longrightarrow G(\mathbb{F}_q^n) \\ (g, W) &\longmapsto gW \end{aligned}$$

where $gW = \{gw \mid w \in W\}$.

$GL_n(\mathbb{F}_q)$ acts on G/B by

$$GL_n(\mathbb{F}_q) \times G/B \longrightarrow G/B$$

$$g(0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n) = (0 \subsetneq gV_1 \subsetneq \cdots \subsetneq V_n).$$

We may extend this to say that $GL_n(\mathbb{F}_q)$ acts on $\mathbb{C}[G/B]$

Since we have just seen that we may 'represent' the Hecke algebra H as elements of $GL_n(\mathbb{F}_q)$, one concludes that H acts $\mathbb{C}[G/B]$.

Theorem If $g \in GL_n(\mathbb{F}_q)$ and $i \in \{1, \dots, n-1\}$ then $gT_i = T_i g$ (as linear transformations of $\mathbb{C}[G/B]$).

The case $\mathbb{F}_1$ (i.e looking at $S(n)$)

In this scenario

$$G/B = \{\text{maximal chains in } S(n)\} \longleftrightarrow \{\text{permutations}\}$$

For example

$$(\emptyset \subsetneq \{\underline{1}\} \subsetneq \{1,\underline{2}\} \subsetneq \{1,2,\underline{3}\}) \longleftrightarrow 123$$

$$(\emptyset \subsetneq \{\underline{1}\} \subsetneq \{1,\underline{3}\} \subsetneq \{1,\underline{2},3\}) \longleftrightarrow 132$$

Then

$$T_i(V_1 \cdots V_n) = \sum_{\substack{V_{i-1} \subsetneq W \subsetneq V_{i+1} \\ W \neq V_i}} (V_1 \cdots V_{i-1} W V_{i+1} \cdots V_n)$$

But, the miracle here is this sum only has one term!

Also, the relations

$$T_j^2 = (1-1)T_j + 1, \qquad T_j T_k = T_k T_j \text{ if } k \notin \{j-1, j+1\}, \qquad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

all hold.

Theorem (Presentation theorem) The symmetric group S_n is generated by

$$s_1, \dots, s_{n-1}$$

with relations

$$s_j^2 = 1, \qquad s_j s_k = s_k s_j \text{ if } k \notin \{j-1, j+1\}, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

This is called the Coxeter presentation.

Proof idea.

(1) Generators A in terms of generators B
(2) Generators B in terms of generators A
(3) Relations A in terms of relations B
(4) Relations B in terms of relations A.