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Advanced Discrete Mathematics
Lecture 10

Let $\mathbb{F}_q$ be a finite field with q elements.

$$GL_n(\mathbb{F}_q) = \{A \in M_{n\times n}(\mathbb{F}_q) \mid \text{there exist } A^{-1} \in M_{n\times n}(\mathbb{F}_q) \text{ s.t } A^{-1}A = I \text{ and } AA^{-1} = I\}.$$

$$\cup\!\!\mid$$

$$B = \{\text{upper triangular matrices in } GL_n(\mathbb{F}_q)\}.$$

For $k \in \{1, \dots, n-1\}$,

$$P_k = \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \in GL_n(\mathbb{F}_q) \;\middle|\; \begin{array}{l} A \in M_{k\times k}(\mathbb{F}_q) \\ B \in M_{k\times n-k}(\mathbb{F}_q) \\ C \in M_{n-k\times n-k}(\mathbb{F}_q) \end{array} \right\}$$

(determinant of the matrix is $\det(A) \times \det(C)$)

$$= \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \;\middle|\; \begin{array}{l} A \in GL_k(\mathbb{F}_q) \\ B \in M_{k\times n-k}(\mathbb{F}_q) \\ C \in GL_{n-k}(\mathbb{F}_q) \end{array} \right\}$$

Remember, these spaces appear as:

- B is the stabiliser of the favourite flag E in $\mathcal{F}(\mathbb{F}_q^n)$
- P_k is the stabiliser of the favourite k-dimensional subspace E_k in $G_k(\mathbb{F}_q^n)$

Remember, also, S_n can be thought of, morally, as

$$S_n = GL_n(\mathbb{F}_1).$$

This is because, morally, we have

$$GL_n(\mathbb{F}_1) = Aut(\mathcal{S}(n)),$$

where $\mathcal{S}(n)$ is the subset lattice of $\{1, \dots, n\}$.

Theorem

(a) $Card(S_n) = n!$

(b) (q-analogue of (a)) $Card(GL_n(\mathbb{F}_q)) = q^{\frac{1}{2}n(n-1)}(q-1)^n [n]!$

(c) $Card(B) = q^{\frac{1}{2}n(n-1)}(q-1)^n$

(d) $Card(P_k) = q^{\frac{1}{2}n(n-1)}(q-1)^n [k]! [n-k]!$

(e) (i) $Card(G/B) = [n]!$ $(= Card(\mathcal{F}(\mathbb{F}_q^n)))$, $G = GL_n(\mathbb{F}_q)$.
(ii) $Card(G/P_k) = \begin{bmatrix} n \\ k \end{bmatrix}$ $(= Card(\mathbb{G}(\mathbb{F}_q^n)_k))$

Proof.

(a) Find $Card(S_n)$.
Recall $S_n = \{n \times n \text{ permutation matrices}\}$.

Choose $w \in S_n$
The number of choices for the first column of w is n
Given column 1, the number of choices for the second column of w is $n-1$
$\vdots$

Given columns $1,\dots,n-1$, the number of choices for the nth column is 1.
So $\mathrm{Card}(S_n) = n\cdot(n-1)\cdots 2\cdot 1 = n!$.

(b) Find $\mathrm{Card}(GL_n(\mathbb{F}_q))$

Recall $GL_n(\mathbb{F}_q) = \{ n\times n \text{ invertible matrices with entries in } \mathbb{F}_q \}$
$= \{ \text{sequences of } n \text{ linearly independant columns} \}$
(where a column is an element of $\mathbb{F}_q^n$)

Choose $g \in GL_n(\mathbb{F}_q)$.
The number of choices for the first column of g is $q^n - 1$.
(This is because $\mathrm{Card}(\mathbb{F}_q^n) = q^n$, but we can't choose 0)
Given column 1, the number of choices for the second column of g is $q^n - q$.
(This is because $\mathrm{Card}(\mathrm{span}\{\text{column } 1\}) = q$, and we can't choose those for the second column.)
Given columns 1 and 2, the number of choices for the third column of g is $q^n - q^2$.
(This is because $\mathrm{Card}(\mathrm{span}\{\text{column } 1, \text{column } 2\} = q^2$, and we can't choose those for the third column.

$\vdots$

Given columns $1,\dots,n-1$, the number of choices for the nth column of g is $q^n - q^{n-1}$.

So

$$\mathrm{Card}(GL_n(\mathbb{F}_q)) = (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$$

$$= (q^n-1)q(q^{n-1}-1)\cdots q^{n-1}(q-1)$$

$$= q^{1+2+\dots+(n-1)} \frac{(q^n-1)}{q-1}\frac{(q^{n-1}-1)}{q-1}\cdots\frac{(q-1)}{q-1}(q-1)^n$$

$$= q^{\frac{1}{2}n(n-1)}(q-1)^n [n]!$$

(At this point, I had a question: 'I have seen these factors before — (in the context of Hall-Littlewood polynomials). What's up with that?
Arun says to look at Macdonald Chapter 2 Section 1.)

(c) Find Card(B).

Recall $B = \left\{ \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ & \ddots & \vdots \\ 0 & & b_{nn} \end{pmatrix} \;\middle|\; b_{ij} \in \mathbb{F}_q,\ \text{matrix is invertible} \right\}$

Choose $b \in B$.

For each diagonal entry b_{ii}, the number of choices for b_{ii} is $q-1$.
(The determinant is product of diagonal entries so we avoid 0)
For each off-diagonal entry b_{ij} with $i \neq j$, the number of choices for b_{ij} is q.

So

$$\text{Card}(B) = (q-1)^{\#\text{diagonal entries}} \cdot q^{\#\text{off-diagonal entries}}$$

$$= (q-1)^n \cdot q^{(n-1)+\cdots+2+1}$$

$$= q^{\frac{1}{2}n(n-1)} (q-1)^n.$$

(d) Find Card(P_k).

Choose $\left(\begin{array}{c|c} A & B \\ \hline & C \end{array}\right)$ with $A \in GL_k(\mathbb{F}_q)$, $B \in M_{k \times n-k}(\mathbb{F}_q)$, $C \in GL_{n-k}(\mathbb{C})$.

The number of choices for A is $q^{\frac{1}{2}k(k-1)}(q-1)^k [k]!$
The number of choices for B is $q^{k(n-k)}$.
The number of choices for C is $q^{\frac{1}{2}(n-k)(n-k-1)}(q-1)^{n-k}[n-k]!$

So $\text{Card}(P_k) = q^{\frac{1}{2}n(n-1)}(q-1)^n [k]! [n-k]!$

(e) We have

$$\mathrm{Card}(G/B) = \frac{\mathrm{Card}(G)}{\mathrm{Card}(B)} = \frac{q^{\frac{1}{2}n(n-1)}(q-1)^n[n]!}{q^{\frac{1}{2}n(n-1)}(q-1)^n}$$

$$= [n]!$$

which is the cardinality of the Flag variety over $\mathbb{F}_q^n$.

Also,

$$\mathrm{Card}(G/P_k) = \frac{\mathrm{Card}(G)}{\mathrm{Card}(P_k)} = \frac{q^{\frac{1}{2}n(n-1)}(q-1)^n[n]!}{q^{\frac{1}{2}n(n-1)}(q-1)^n[k]![n-k]!}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix}$$

which is the cardinality of the Grassmanian of k-planes in n-space over $\mathbb{F}_q$.

A consequence of this is the following:

$$\mathrm{Card}(\mathbb{P}^{n-1}(\mathbb{F}_q)) = \mathrm{Card}(\mathbb{G}(\mathbb{F}_q^n)_1)$$

$$= \mathrm{Card}(G/P_1)$$

$$= \begin{bmatrix} n \\ 1 \end{bmatrix}$$

$$= [n]$$

$$= 1 + q + \cdots + q^{n-1}$$

Another consequence is:

$$\begin{aligned}
\mathrm{Card}(\mathbb{P}^1(\mathbb{F}_q)) &= \mathrm{Card}(G(\mathbb{F}_q^2, 1)) \\
&= \mathrm{Card}\left(\frac{GL_2(\mathbb{F}_q)}{P_1}\right) \\
&= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\
&= [2] \\
&= 1 + q
\end{aligned}$$