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Advanced Discrete Mathematics
Lecture 9

Projective space and cosets

Let $\mathbb{F}$ be a field. Recall

$$\mathbb{F}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \middle| \; a_i \in \mathbb{F} \right\} \quad \text{and} \quad \mathbb{F}^\times = \mathbb{F} - \{0\}.$$

Define an equivalence relation on $\mathbb{F}^n - \{0\}$ by

$$[a_1, \ldots, a_n] = [\lambda a_1, \ldots, \lambda a_n], \quad \text{if} \quad \lambda \in \mathbb{F}^\times.$$

Alternatively,

$$\operatorname{span}\left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix} \right\}, \quad \text{if} \quad \lambda \in \mathbb{F}^\times.$$

Recall that

$$G(\mathbb{F}^n) = \{ \mathbb{F}\text{-subspaces of } \mathbb{F}^n \}, \text{ ordered by inclusion}$$

$$= \bigsqcup_{k=0}^{n} G(\mathbb{F}^n)_k,$$

where

$$G(\mathbb{F}^n)_k = \{ \mathbb{F}\text{-subspaces of } \mathbb{F}^n \text{ with } \mathbb{F}\text{-dimension } k \}.$$

The projective space $\mathbb{P}^{n-1}$ is

$$\mathbb{P}^{n-1} = \{\text{equivalence classes of } \mathbb{F}^n \text{ under the equivalence relation defined on the first page}\}$$

Or, alternatively,

$$\mathbb{P}^{n-1} = \mathbb{G}(\mathbb{F}^n)_1.$$

The Grassmanian of k-planes in n-space is $\mathbb{G}(\mathbb{F}_n)_k$.

The flag variety is

$$\mathcal{F}(\mathbb{F}^n) = \{\text{maximal chains in } \mathbb{G}(\mathbb{F}^n)\}$$

(Projective space is level 1, Grassmanian is level k, and the flag variety is the chains.)

Let $G = GL_n(\mathbb{F})$. Define

$$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix} \right\} \leq G \quad \left(\text{upper triangular matrices of size } n \times n \text{ and invertible}\right)$$

$$P_k = \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \in G \;\middle|\; \begin{array}{ll} A \in M_{k \times k}(\mathbb{F}) & B \in M_{k \times n-k}(\mathbb{F}) \\ C \in M_{n-k \times n-k}(\mathbb{F}) & \end{array} \right\} \leq G.$$

for $k \in \{1, \dots, n-1\}$.

B and $P_1, \dots, P_{n-1}$ are subgroups of G (but are not normal!)

Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the favourite basis of $\mathbb{F}^n$.

Let $E_k = \operatorname{span}\{\varepsilon_1, \dots, \varepsilon_k\}$ be the favourite k-subspace of $\mathbb{F}^n$.

Let $E = (\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq \mathbb{F}^n)$ be the favourite maximal chain of of $\mathbb{F}^n$.

Interlude : who cares?

Answer: $\mathcal{G}(\mathbb{F}^n)$ has LOTS of symmetries.
In fact,

$$\operatorname{Aut}(\mathcal{G}(\mathbb{F}^n)) = PGL_n(\mathbb{F})$$

where

$$PGL_n(\mathbb{F}) = \frac{GL_n(\mathbb{F})}{Z}, \qquad Z = \left\{ \begin{pmatrix} c & & \\ & \ddots & \\ & & c \end{pmatrix} \in GL_n(\mathbb{F}) \;\middle|\; c \in \mathbb{F} \right\}$$

In other words, there is an action of $G = GL_n(\mathbb{F})$ on $\mathcal{G}(\mathbb{F}^n)$:

$$\begin{aligned} G \times \mathcal{G}(\mathbb{F}^n) &\longrightarrow \mathcal{G}(\mathbb{F}^n) \\ (g, W) &\longmapsto gW \end{aligned}$$

where

$$gW = \left\{ g \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} \;\middle|\; \begin{pmatrix} w_1 \\ \vdots \\ w_k \end{pmatrix} \in W \right\}.$$

Notice that the constant matrices $\begin{pmatrix} c & & \\ & \ddots & \\ & & c \end{pmatrix}$, that is elements of Z, are acting as the identity on $\mathcal{G}(\mathbb{F}^n)$.

Another way to say this is

$$\mathrm{Stab}_G(\mathcal{G}(\mathbb{F}^n)) = Z.$$

We have that the orbit of E_1 under the G action is

$$G \cdot E_1 = \mathbb{P}^{n+1} = \mathcal{G}(\mathbb{F}^n)_1.$$

The orbit of E_k under the G action is

$$G \cdot E_k = \mathcal{G}(\mathbb{F}^n)_k.$$

The result is the following:

(a) $\mathrm{Stab}_G(E_k) = P_k$ (maximal standard parabolic subspaces)

(b) $\mathrm{Stab}_G(E) = B$ (Borel subgroup)

So, by orbit-stabiliser:

(c) $G/P_k \simeq G \cdot E_k = \mathcal{G}(\mathbb{F}^n)_k$ (as G-sets)

(d) $G/B \simeq G \cdot E = \mathcal{F}(\mathbb{F}^n)$ (as G-sets)

The image is:

$$G \supseteq P_k \supseteq B \supseteq Z$$

Maxims: MAXIMAL CHAINS GIVE MORE REFINED INFORMATION THAN THE GRASSMANIAN BUT IT IS STILL VERY CONTROLLED.

Why is it very controlled?

Theorem (Bruhat Decomposition) Let $G = GL_n(\mathbb{F})$ and $W = S_n$. Then,

$$G = \bigsqcup_{w \in W} BwB \quad , \quad \text{where } BwB = \{b_1 w b_2 \mid b_1, b_2 \in B\}.$$

horrible! $\qquad$ controlled!

(EVERY HORRIBLE THING CAN BE WRITTEN IN A CONTROLLED WAY)

Theorem (Counting) Let $\mathbb{F}_q$ be the field with q elements.

(a) $\mathrm{Card}(GL_n(\mathbb{F}_q)) = [n]!\,(q-1)^n\, q^{\frac{n(n-1)}{2}}$

(b) $\mathrm{Card}(B) = q^{\frac{n(n-1)}{2}} (q-1)^n$

(c) $\mathrm{Card}(\mathcal{F}(\mathbb{F}_q^n)) = \mathrm{Card}(G/B) = [n]!$

(d) $\mathrm{Card}(P_k) = q^{\frac{n(n-1)}{2}} (q-1)^n\, [k]!\,[n-k]!$

(e) $\mathrm{Card}(G(\mathbb{F}^n)_k) = \mathrm{Card}(G/P_k) = \begin{bmatrix} n \\ k \end{bmatrix}.$

(f) $\mathrm{Card}(GL_n(\mathbb{F}_1^n)) = \mathrm{Card}(S_n) = n!$

(g) $\mathrm{Card}(B(\mathbb{F}_1^n)) = \mathrm{Card}(\{1\}) = 1$

(h) $\mathrm{Card}(\mathcal{F}(\mathbb{F}_1^n)) = \mathrm{Card}(S_n/\{1\}) = n!$