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Advanced Discrete Mathematics
Problem Solving 8

Coxeter System

A Coxeter system is $M \in M_n(\mathbb{Z}_{>0})$ such that if $i, j \in \{1, \dots, n\}$ and $i \neq j$ then

(a) $m_{ij} = m_{ji}$
(b) $M_{ii} = 1$

The Coxeter group W of M is the group generated by $s_1, \dots, s_n$ with relations $(s_i s_j)^{m_{ij}} = 1$ for all $i, j \in \{1, \dots, n\}$.

Example 1: $M = (1)$ then $W = \{1, s_1\}$ with $s_1^2 = 1$.

Example 2: $M = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ then W is generated by s_1 and s_2

with relations: $s_1^2 = 1$, $s_2^2 = 1$, $(s_1 s_2)^3 = 1$.

Yet $(s_1 s_2)^3 = 1$

$\Leftrightarrow s_1 s_2 s_1 s_2 s_1 s_2 = 1$

$\Leftrightarrow s_1 s_2 s_1 s_2 s_1 = s_2$

$\Leftrightarrow s_1 s_2 s_1 s_2 = s_2 s_1$

$\Leftrightarrow s_1 s_2 s_1 = s_2 s_1 s_2$

Example 3: $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ then W is generated by s_1, s_2 and s_3

with relations: $s_1^2 = 1$, $s_2^2 = 1$, $s_3^2 = 1$, $(s_1 s_2)^3 = 1$, $(s_2 s_3)^3 = 1$ and $(s_1 s_3)^2 = 1$.

Yet $(s_1 s_3)^2 = 1$

$\Leftrightarrow s_1 s_3 s_1 s_3 = 1$

$\Leftrightarrow s_1 s_3 s_1 = s_3$

$\Leftrightarrow s_1 s_3 = s_3 s_1$

Even though Coxeter systems were quite important historically, Arun believes that Cartan matrices are the way to go as it reveals better insights into the Lie algebraic foundations of the underlying groups.

Historically (approximately) we have:

1. Cartan matrix 1875 (Lie algebras)

2. Coxeter 1930 (Group theory via generators and relations)

3. Kac-Moody 1968 (Back to Lie algebras)

Good reference: Reflection Groups and Invariant Theory by Richard Kane.

Quantum Groups

A quantum group is not a group and is not quantum. This is just high-quality marketing.

classical $\begin{cases} \text{fin-dim vector spaces} \\ \text{linear transformations} \end{cases}$ quantum $\begin{cases} \{\text{functions}\} \text{ is inf-dim vector space} \\ \text{linear operators} \end{cases}$

Cayley Graphs

Let G be a group and $\{g_1, \ldots, g_i\}$ be a set of generators. The Cayley graph of G with generators $\{g_1, \ldots, g_i\}$ is the graph with

Vertices: G Edges: $g \xrightarrow{g_i} h$ if $h = g_i g$.

The power of Cayley graphs comes from the fact that it gives a pictorial way to understand the group.

Let $n = |G|$. For $i \in \{1, \dots, l\}$ define

$$G_i \in M_n(\{0,1\}) \quad \text{with} \quad G_{gh} = \begin{cases} 1, & \text{if } g \xrightarrow{g_i} h \\ 0, & \text{otherwise} \end{cases}$$

<u>Fano plane example</u>

We have $\mathbb{G}(\mathbb{F}_2^3) = \{ \mathbb{F}_2\text{-subspaces of } \mathbb{F}_2^3 \}$

$$\mathbb{F}_2^3 = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \middle| \; a_i \in \mathbb{F}_2 \right\} \qquad \mathbb{F}_2 = \{0,1\}$$

The Hasse diagram for $\mathbb{G}(\mathbb{F}_2^3)$ is

$$\mathbb{F}_2^3 \qquad 1$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad 7$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad 7$$

$$\{0\} \qquad 1$$

Notice that $\begin{bmatrix} 3 \\ 0 \end{bmatrix}_2 = 1, \begin{bmatrix} 3 \\ 1 \end{bmatrix}_2 = 7, \begin{bmatrix} 3 \\ 2 \end{bmatrix}_2 = 7, \begin{bmatrix} 3 \\ 3 \end{bmatrix}_2 = 1$