Gypsy Akhyar
14th August 2025

# Advanced Discrete Mathematics
## Lecture 8

### Automorphisms of posets

A morphism of posets from $P$ to $Q$ is a function

$$f : P \to Q$$

such that

if $x, y \in P$ and $x \leq y$ then $f(x) \leq f(y)$.

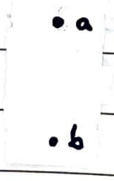
An isomorphism of posets from $P$ to $Q$ is a morphism of posets

$$f : P \to Q$$

such that:

(a) the inverse $f^{-1} : Q \to P$ exists
(b) $f^{-1}$ is a morphism of posets.

Note: if you tried to define an isomorphism of posets as a bijective morphism of posets, it would not work.

The example is $P =$ 

and $Q =$ 


So $P$ has no relations and $Q$ has $c \leq d$.
One should check $f : P \to Q$, $a \mapsto c$, $b \mapsto d$ is a morphism that is bijective but not an isomorphism

Let $P$ be a poset. An automorphism of $P$ is an isomorphism of posets $f : P \to P$.

The automorphism group of $P$ is

$$\mathrm{Aut}(P) = \{\text{automorphisms of } P\}$$

with the operation of composition of functions.

Recall that

$(1+x)^n = \sum_{k=0}^{n} x^k \binom{n}{k}$ is the rank generating function for $\mathcal{S}(n)$

(this is the binomial theorem)

$$(1+x)(1+xq)\cdots(1+xq^{n-1}) = \sum_{k=0}^{n} x^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}$$

is the 'adjusted' rank generating function for $\mathcal{G}(\mathbb{F}_q^n)$.
(this is the $q$-binomial theorem)

We notice that the $q$ approaches 1 limit 'degenerates' the $q$-binomial theorem to the ordinary binomial theorem

In this way, philosophically, $\mathcal{S}(n) = \mathcal{G}(\mathbb{F}_1^n)$.

<u>Theorem</u> Let $\mathbb{F}$ be a field.

(a) $\mathrm{Aut}(\mathcal{G}(\mathbb{F}^n)) = GL_n(\mathbb{F})$

(b) $\mathrm{Aut}(\mathcal{S}(n)) = S_n$

## Proof idea

$GL_n(\mathbb{F}) = Aut(\mathbb{F}^n)$ (in the category of vector spaces)

Since $\mathcal{G}(\mathbb{F}^n)$ 'comes from' $\mathbb{F}^n$, then we should expect a map

$$Aut(\mathbb{F}^n) \longrightarrow Aut(\mathcal{G}(\mathbb{F}^n))$$

We will define this map in the following way.

If $W \subseteq \mathbb{F}^n$ is a subspace and $g \in Aut(\mathbb{F}^n)$ then

$$gW = \{gw \mid w \in W\} \in \mathcal{G}(\mathbb{F}^n)$$ (recall $gW \in \mathcal{G}(\mathbb{F}^n)$ just means that $gW$ is a subspace)

So multiplying by $g \in Aut(\mathbb{F}^n)$ is giving a map $\mathcal{G}(\mathbb{F}^n) \to \mathcal{G}(\mathbb{F}^n)$.
It remains to check the poset condition.

If $Z \subseteq W \subseteq \mathbb{F}^n$ are subspaces, then

$$gZ \subseteq gW$$

and so we are good.

What you must now show is that this map is an isomorphism of groups. $\square$

## Time for a tangent!

The flag variety is

$$\mathcal{F}(\mathbb{F}^n) = \{\text{maximal chains in } \mathcal{G}(\mathbb{F}^n)\}$$

A maximal chain is a path from $\{0\}$ to $\mathbb{F}^n$ in the Hasse diagram of $\mathbb{G}(\mathbb{F}^n)$.

$$\mathcal{F}(\mathbb{F}_1^n) = \{\text{maximal chains in } \mathbb{S}(n)\}$$

For example

$$\mathcal{F}(\mathbb{F}_1^3) = \begin{Bmatrix} 123 & 132 \\ 213 & 231 \\ 312 & 321 \end{Bmatrix}$$

where

$ijk$ is shorthand for $\varnothing - \{i\} - \{i,j\} - \{i,j,k\}$.

So, in fact

$$\mathcal{F}(\mathbb{F}_1^n) = \{\text{permutations of } \{1,\ldots,n\}\}$$

and thus

$$\mathrm{Card}(\mathcal{F}(\mathbb{F}_1^n)) = n!$$

Next, consider

$$\mathcal{F}(\mathbb{F}_q^n) = \{\text{maximal chains in } \mathbb{G}(\mathbb{F}_q^n)\}$$

or, in more common parlance

$$\mathcal{F}(\mathbb{F}_q^n) = \{\text{flags in } \mathbb{F}_q^n\}.$$

**Proposition (Bruhat decomposition)**

$$\mathrm{Card}(\mathcal{F}(\mathbb{F}_q^n)) = [n]!$$

$\mathbb{F}^n$ has a favourite ordered $\mathbb{F}$-basis

$(\varepsilon_1, \dots, \varepsilon_n)$, where $\varepsilon_i = (0 \cdots \overbrace{1}^{i^{th}\text{ position}} \cdots 0)$

The base point of $\mathcal{F}(\mathbb{F}^n)$, or the standard flag, is the flag

$$E = (\{0\} = E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_{n-1} \subsetneq E_n = \mathbb{F}^n)$$

where $E_k = \mathbb{F}\text{-span}\{\varepsilon_1, \dots, \varepsilon_k\}$.

Let $g \in GL_n(\mathbb{F})$.
Then,

$$gE = (gE_0 \subsetneq \cdots \subsetneq gE_n)$$

$$= (\{0\} \subseteq gE_1 \subsetneq \cdots \subsetneq gE_{n-1} \subsetneq \mathbb{F}^n) \in \mathcal{F}(\mathbb{F}^n).$$

More precisely, we have an action of $GL_n(\mathbb{F})$ on $\mathcal{F}(\mathbb{F}^n)$.
Let $E \in \mathcal{F}(\mathbb{F}^n)$ and define

$$B = \{g \in GL_n(\mathbb{F}) \mid gE = E\} \left( = \mathrm{Stab}_{GL_n(\mathbb{F})}(E) \right)$$

**Proposition**

$$B = \{g \in GL_n(\mathbb{F}) \mid g \text{ is upper triangular}\}$$