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Advanced Discrete Mathematics
Lecture 6

Posets and Lattices

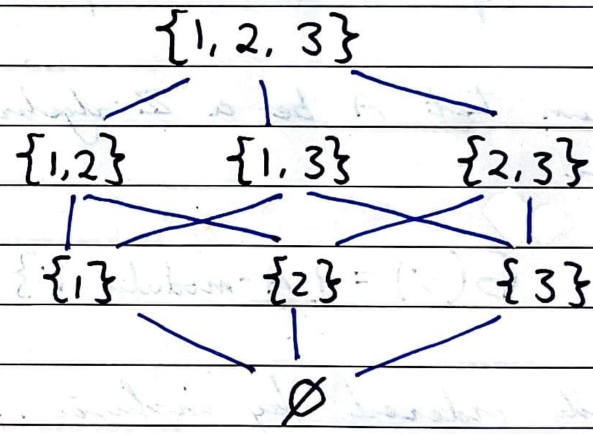
Two examples:

(1) Let $n \in \mathbb{Z}_{>0}$. The <u>subset lattice</u> is

$$S(n) = \{\text{subsets of } \{1,\dots,n\}\}$$

partially ordered by inclusion.

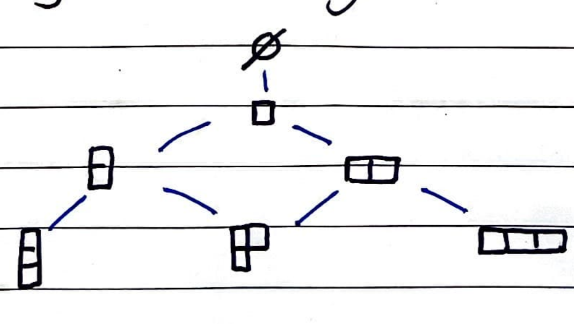
For example: $S(3)$:



(2) The <u>Young lattice</u> is

$$\mathbb{Y} = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Y}_n = \{\text{partitions}\}$$

partially ordered by inclusion (where we think of λ as a set of boxes)



We have that, looking at example (1),

$$\mathcal{S}(n) = \bigsqcup_{k=0}^{n} \mathcal{S}(n)_k \quad \text{where} \quad \mathcal{S}(n)_k = \left\{ \begin{matrix} \text{subsets of } \{1,\ldots,n\} \\ \text{with cardinality } k \end{matrix} \right\}$$

and $\quad \mathrm{Card}(\mathcal{S}(n)_k) = \binom{n}{k}$.

Let S be a set.
A relation on S is a subset $\leq$ of $S \times S$.
We write

$$x \leq y \qquad \text{if} \qquad (x,y) \in \leq .$$

A poset, or partially ordered set, is the set P with a relation $\leq$ on P such that

(a) If $x \in P$ then $x \leq x$

(b) If $x,y,z \in P$ and $x \leq y$ and $y \leq z$ then $x \leq z$

(c) If $x,y \in P$ and $x \leq y$ and $y \leq x$ then $x = y$.

The Hasse diagram of a poset P is the graph with

vertices P and $x \to y$ if $x < y$ and there does not exist $z \in P$ with $x < z < y$

where we have used the notation $x < y$ to mean

$$x \leq y \qquad \text{and} \qquad x \neq y$$

A maximal chain in P is a sequence $(x_1, \dots, x_n)$ in P with $n \in \mathbb{Z}_{>0}$ such that

(a) If $i \in \{1, \dots, n-1\}$ then $x_i < x_{i+1}$ and there does not exist $z \in P$ with $x_i < z < x_{i+1}$.

(b) There does not exist $z \in P$ such that $x_n < z$

(c) There does not exist $z \in P$ such that $z < x_1$

Let P be a poset and $E \subseteq P$.
The infimum, or greatest lower bound, of E in P is $l \in P$ such that

(a) If $e \in E$ then $l \le e$

(b) If $m \in P$ such that m satisfies: If $e \in E$ then $m \le e$, then $m \le l$.

The supremum, or least upper bound, of E in P is $\gamma \in P$ such that

(a) If $e \in E$ then $e \le \gamma$

(b) If $\tau \in P$ such that τ satisfies: If $e \in E$ then $e \le \tau$ then $\gamma \le \tau$

Notation: If $k \in \mathbb{Z}_{>0}$ and $x_1, \dots, x_n \in P$ then

$$\inf(x_1, \dots, x_n) = \inf(\{x_1, \dots, x_n\}) \quad \text{and} \quad \sup(x_1, \dots, x_n) = \sup(\{x_1, \dots, x_n\})$$

A <u>lattice</u> is a poset P such that if $x, y \in P$ then $\inf(x,y)$ and $\sup(x,y)$ exist in P.

Let P be a lattice.
Use the notation

$$x \wedge y = \inf(x,y) \qquad \text{and} \qquad x \vee y = \sup(x,y)$$

x meet y $\qquad\qquad$ x join y

A <u>modular lattice</u> is a lattice P such that

if $m, n, p \in P$ and $m \leq p$ then $m \vee (n \wedge p) = (m \vee n) \wedge p$

ring $\qquad\qquad$ vector space

<u>Theorem</u> Let A be a $\mathbb{Z}$-algebra and let V be an A-module
Define

$$\mathcal{G}(V) = \{A\text{-modules } V\}$$

partially ordered by inclusion.

Then, $\mathcal{G}(V)$ is a modular lattice.