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Advanced Discrete Mathematics
Problem Solving 5

Referencing

Arun uses 'mathscinet' style

this is arxiv year

\bibitem[N22]{N22} F. Name, \emph{Title of paper},
J. of interest \textbf{52} (2024) 63-79, MR 562873,
arxiv: 2206.5871

↳ volume number

For a bibliography in this style, check out a paper published by AMS.

Alternatively, many will use bibTeX.

Tenents of discrete mathematics

Posets and Counting	Generating functions	Symmetry

Of course, these three often interact and intersect.

Writing

For writing expositions, a key consideration is to keep

focus!

This means you have to throw away a lot of stuff.

(Maxim: Give both an illustrative example and non-example).

Orbits and Stabilisers

Let G be a group.
A G-set is a set X with a function $G \times X \to X$, called a G-action, such that

(a) If $x \in X$ then $1x = x$
(b) If $g_1, g_2 \in G$ and $x \in X$ then $g_1(g_2 x) = (g_1 g_2)x$

(Analogue of image)
Let X be a G-set and $x \in X$.
The orbit of x is $Gx = \{gx \mid g \in G\}$

(Analogue of kernel)
The stabiliser of x is $\mathrm{Stab}_G(x) = \{h \in G \mid hx = x\}$

Let X, Y be G-sets.
A morphism of G-sets is a function $f: X \to Y$ such that

(a) If $g \in G$ and $x \in X$ then $g f(x) = f(gx)$.

If H is a subgroup of G (note that H need not be normal) then

(a) $G/H = \{gH \mid g \in G\}$ where $gH = \{gh \mid h \in H\}$ is the set of cosets of H in G.

(b) G/H is a G-set with G-action given by $g(xH) = (gx)H$.

Proposition As G-sets,

$$Gx \simeq G/\mathrm{Stab}_G(x).$$

In particular, taking cardinalities, we have

$$\mathrm{Card}(G) = \mathrm{Card}(Gx)\,\mathrm{Card}(\mathrm{Stab}_G(x)).$$

One uses this theorem, for example, to show the correspondence

$$\left\{\begin{matrix}\text{partitions}\\ \text{of } n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{conjugacy classes}\\ \text{in } S_n\end{matrix}\right\}$$

Loop Grassmanian

The *loop Grassmanian* is G/K where

$$G = GL_n(\mathbb{F}((\varepsilon))) \qquad \text{and} \qquad K = GL_n(\mathbb{F}[[\varepsilon]])$$

$\uparrow$ which is kind of like $GL_n(\mathbb{Q})$

$\uparrow$ which is kind of like $GL_n(\mathbb{Z})$

where $\mathbb{F}$ is a field and

$$\mathbb{F}[[\varepsilon]] = \{a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots \mid a_i \in \mathbb{F}\}$$

and

$$\begin{aligned}\mathbb{F}((\varepsilon)) &= \text{field of fractions of } \mathbb{F}[[\varepsilon]] \\ &= \left\{ \frac{a(\varepsilon)}{b(\varepsilon)} \;\middle|\; \begin{matrix} a(\varepsilon), b(\varepsilon) \in \mathbb{F}[[\varepsilon]] \\ \text{and } b(\varepsilon) \neq 0 \end{matrix} \right\}\end{aligned}$$

with $\frac{a(\varepsilon)}{b(\varepsilon)} = \frac{c(\varepsilon)}{d(\varepsilon)}$ if $a(\varepsilon)d(\varepsilon) = b(\varepsilon)c(\varepsilon)$.

The Grassmanian is $Gr_{k,n} = G/P_k$ where

$G = GL_n(\mathbb{C})$ and $P_k = \left\{ \begin{pmatrix} \underbrace{\begin{array}{cc} k\{ & \square \\ n-k\{ & \square \end{array}}_{\text{upper triangular matrices with a } k\times k \text{ and } n-k\times n-k \text{ block}} & * \end{pmatrix} \in GL_n(\mathbb{C}) \right\}$