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Advanced Discrete Mathematics
Problem Solving 4

On functions

Motivation for functions: they are morphisms in the category of sets

Motivation for formulation for the definition of function:

how do we execute the proofs we need to do involving functions?

Fundamental results:

(0) Composition is an operation on functions

(1) A function has an inverse function if and only if it is bijective.

(2) Every function $f: S \to T$ can be expressed as a composition $f = g \circ h$ with h injective and g surjective

change this statement to a non-existential statement

Let $f: S \to T$ be a function.
Define $X = \{ f^{-1}(t) \mid t \in T \}$ where $f^{-1}(t) = \{ s \in S \mid f(s) = t \}$
Define $g: X \to T$ $\quad$ $h: S \to X$ such that
$g(x) = t$ if $x = f^{-1}(t)$ and $h(s) = f^{-1}(f(s))$.
Then, $f = g \circ h$, g is surjective and f is injective

Sketch of the proof that $\sum_{\lambda \in \mathbb{B}_n} b_\lambda^2 = (2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$

(1) The irreducible $B_n(x)$-modules are indexed by $\lambda \in \mathbb{B}_n$ (where $B_n(x)$ is the Brauer algebra)

(2) The edges in $\mathbb{B}_n$ give the decomposition of the induced modules $\mathrm{Ind}_{B_n(x)}^{B_{n+1}(x)}(B_n^\lambda)$.

(3) So, $\dim(B_n^\lambda) = \#$ paths from $\emptyset$ to λ in $\mathbb{B}_n$.

(4) Since $B_n(x)$ is semisimple with dimension $(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1$, then the result follows from the Artin–Wedderburn theorem for semisimple algebras.

Writing

The words 'isomorphic' and 'subrepresentation' are indicator words for the existence of a category.

A category $\mathcal{C}$ is a collection of data which includes

(a) Objects: $x \in \mathcal{C}$
(b) Morphisms: $X \xrightarrow{f} Y$ (where $X, Y \in \mathcal{C}$).

Subobjects of X are pairs (Z, ι) with $\iota: Z \to X$ which (in our minds) are injective. (However, to write this properly we have to say this slightly differently).

Isomorphism are morphisms $X \xrightarrow{f} Y$ for which there exists an inverse morphism $Y \xrightarrow{f^{-1}} X$ which is a morphism.