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Advanced Discrete Mathematics
Lecture 4

Partitions

A partition $\lambda = (\lambda_1, \dots, \lambda_l)$ where $l \in \mathbb{Z}_{\geq 0}$ and $\lambda_1, \dots, \lambda_l \in \mathbb{Z}_{\geq 0}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$.

A box is an element of $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(r,c) \mid r,c \in \mathbb{Z}\}$

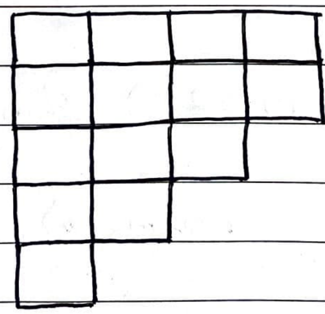
Identify a partition $\lambda = (\lambda_1, \dots, \lambda_l)$ with a set of boxes

$$\lambda = \{(r,c) \mid r \in \{1, \dots, l\},\ c \in \{1, \dots, \lambda_r\}\}$$

so that λ_r boxes in row r.

For example

$$\lambda = (4\ 4\ 3\ 2\ 1) =$$



$$= \left\{ \begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & \\ (4,1) & (4,2) & & \\ (5,1) & & & \end{matrix} \right\}$$

Suppose $\lambda = (\lambda_1, \ldots, \lambda_l)$ is a partition.

(a) The length of λ is $l(\lambda) = l$;

(b) The weight of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_l$.

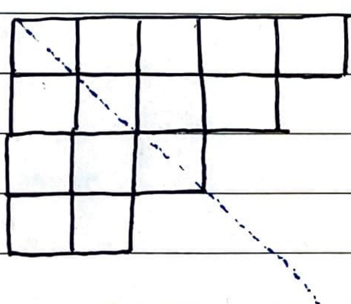
For example, if $\lambda = (44321)$ then $l(\lambda) = 5$ and $|\lambda| = 14$.

Write $\mu \subseteq \lambda$ is μ is a subset of λ (as sets of boxes)

The conjugate of a partition λ is denoted

$$\lambda' = \{ (c,r) \mid (r,c) \in \lambda \}.$$

For example, if $\lambda = (44321)$, then

$\lambda' =$  $= (5432)$

which can be seen as reflecting λ over $y = x$.

Let $\mathbb{Y}_n = \{ \text{partitions } \lambda \mid |\lambda| = n \}$

$= \{ \text{partitions with } n \text{ boxes} \}$

and $\mathbb{Y} = \bigsqcup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Y}_n$ (the set of all partitions)

Let $n \in \mathbb{Z}_{\geq 0}$ and $\lambda \in \mathcal{Y}_n$.

A standard tableau of shape λ is a function

$$T : \lambda \longrightarrow \{1, \ldots, n\}$$

such that

(a) If $(r,c), (r,c+1) \in \lambda$, then $T(r,c) < T(r,c+1)$
(this means that we are increasing from left to right)

(b) If $(r,c), (r+1,c) \in \lambda$, then $T(r,c) < T(r+1,c)$.
(this means we are increasing from top to bottom)

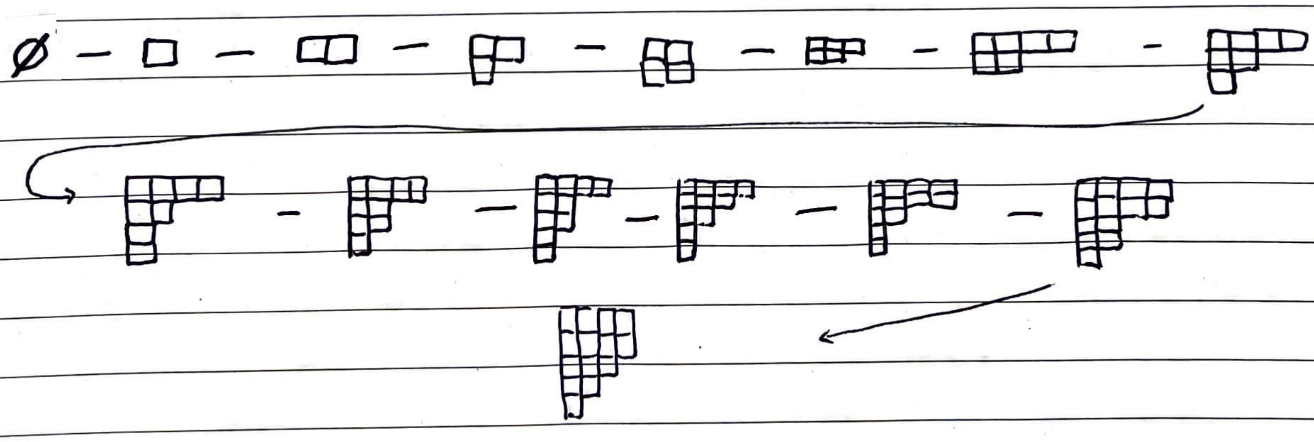
For example, if $\lambda = (44321)$, then

$T =$

1	2	5	6
3	4	11	12
7	9	14	
8	13		
10			

A standard tableau of shape λ is in fact a walk from $\emptyset$ to λ.

For the above example, we have



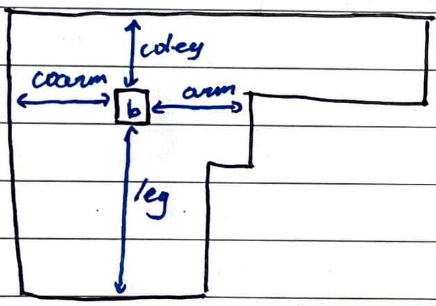
Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition.
Define

$$f_\lambda = \mathrm{Card}\{\text{standard tableaux of shape } \lambda\}$$
$$= \mathrm{Card}\{\text{paths from } \emptyset \text{ to } \lambda \text{ in } \mathbb{Y}\}$$

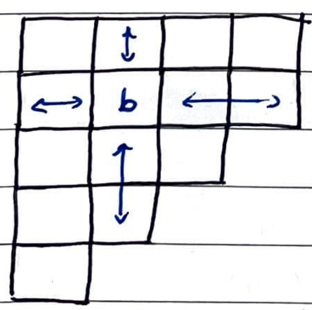
Let $(r,c) \in \lambda$.
Define

$$\mathrm{arm}_\lambda(r,c) = \lambda_r - c \qquad \mathrm{coarm}_\lambda(r,c) = c-1$$
$$\mathrm{leg}_\lambda(r,c) = \lambda'_c - r \qquad \mathrm{coleg}_\lambda(r,c) = r-1$$

The picture is



For example if $\lambda = (44321)$ and $b = (2,2)$ then

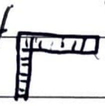


$$\mathrm{coarm}_\lambda(b) = 1$$
$$\mathrm{arm}_\lambda(b) = 2$$
$$\mathrm{coleg}_\lambda(b) = 1$$
$$\mathrm{leg}_\lambda(b) = 2$$

<u>Theorem</u> Let λ be a partition and $|\lambda| = n$.

(a) $$f_\lambda = \frac{n!}{\prod_{b \in \lambda} (\mathrm{arm}_\lambda(b) + \mathrm{leg}_\lambda(b) + 1)}$$

(b) $$\sum_{\lambda \in Y_n} f_\lambda^2 = n!$$

(Remark: (a) is a generalisation of the theorem that $f_{(k, \underbrace{1, 1, \dots, 1, 1}_{n-k})} = f$  $= \binom{n}{k}$.)

There are many proofs of (a), which include

- Frame-Robinson-Thrall (generalising Pascal's triangle)
- Littlewood (relating to Schur polynomials)
- Greene-Nijenhuis-Wilf (probabilistic)
- Peterson (Schubert Calculus)
- Nakada (Arun's choice if you want to learn)

Part (b) can be proved using the Robinson-Schensted-Knuth correspondence (or, in modern math language this is in fact the theory of crystals).