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Advanced Discrete Mathematics
Problem Solving / Questions 2

Representation of Brauer algebra

A representation of an algebra A is an A-module.

An A-module is a vector space M with an action of A.

An action of A on M is a function

$$\begin{aligned} A \times M &\longrightarrow M \\ (a, m) &\longmapsto am \end{aligned}$$ such that

(a) If $a_1, a_2 \in A$ and $m \in M$, then $a_1(a_2 m) = (a_1 a_2)m$
(b) If $m \in M$, then $1m = m$
(c) If $a \in A$ and $m_1, m_2 \in M$, then $a(m_1 + m_2) = am_1 + am_2$
(d) If $a_1, a_2 \in A$ and $m \in M$, then $(a_1 + a_2)m = a_1 m + a_2 m$

An algebra is a vector space A with multiplication (that satisfies some nice properties).

This multiplication in A makes A into an A-module
So, this is a representation of A called the regular representation).

This makes $a \in A$ an operator on M (an operator, in this context, is just a linear transformation).

<u>If $n \in \mathbb{Z}_{>0}$, what is $(x+y)^n$?</u>

$\mathbb{Z}_{>0}$ is, a priori, a set:

$$\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}.$$

In fact, 2, 3, 4, 5 ... are not 'real'. they are:

$$2 = 1 + 1$$
$$3 = 1 + 1 + 1$$
$$4 = 1 + 1 + 1 + 1$$
$$5 = 1 + 1 + 1 + 1 + 1$$
$$\vdots$$

So, in fact, $\mathbb{Z}_{>0}$ is just '1' and '+'.

The fancy way to say '1' and '+' is:

a free monoid generated by 1 with operation $+$.

Proof by induction is not a proof method. Proof by induction is a name for the definition of $\mathbb{Z}_{>0}$.

Base case $\longleftrightarrow$ 1

Induction step $\longleftrightarrow$ $(n+1) = (n) + 1$.

Where does multiplication come from?

I would like a function $\varphi : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ that respects $+$.
This means that if $a, b \in \mathbb{Z}_{\geq 0}$, then

$$\varphi(a+b) = \varphi(a) + \varphi(b).$$

$\mathbb{Z}_{\geq 0}$ is, remember, just 1 and $+$.
Suppose that $\varphi(1) = m$, where $m \in \mathbb{Z}_{\geq 0}$.
Then, respecting $+$ means that

$$\varphi(2) = \varphi(1+1) = \varphi(1) + \varphi(1) = m + m = 2m$$
$$\varphi(3) = \varphi(1+1+1) = \varphi(1) + \varphi(1) + \varphi(1) = m+m+m = 3m$$
$$\vdots$$

So, the whole function φ was completely determined by the image of 1. This unique map for each $m \in \mathbb{Z}_{\geq 0}$ is commonly called 'multiplication by m'.

So, induction is extra powerful as it has multiplication baked in.

Back to $(x+y)^n$

$(x+y)^n \in \mathbb{C}[x,y]$, which is the commutative $\mathbb{C}$-algebra generated by x and y.

$\mathbb{C}[x,y]$ has distributive laws, which means that we should evaluate $(x+y)^n$ by distributive laws.

$$(x+y)(x+y) = x(x+y) + y(x+y)$$
$$= xx + xy + yx + yy$$
$$= x^2 + xy + xy + y^2$$
$$= x^2 + (1+1)\, xy + y^2$$
$$= x^2 + 2xy + y^2 .$$

A polynomial $p \in \mathbb{C}[x,y]$ is $p(x,y) = \sum_{k,l \in \mathbb{Z}_{\geq 0}} A_{k,l}\, x^k y^l$

That is, $p(x,y) = A_{00}\, x^0 y^0 + A_{10}\, x^1 y^0 + A_{01}\, x^0 y^1 + A_{11}\, x^1 y^1 + \cdots$

So, the question we are really asking is if $n \in \mathbb{Z}_{\geq 0}$ and

$$(x+y)^n = \sum_{k,l \in \mathbb{Z}_{\geq 0}} A^{(n)}_{k,l}\, x^k y^l ,$$

then what are $A^{(n)}_{k,l}$?

Base case $(n=1)$ gives $A^{(1)}_{10} = 1$, $A^{(1)}_{01} = 1$ and $A^{(1)}_{k,l} = 0$ otherwise.

Base case $(n=2)$ gives $A^{(2)}_{20} = 1$, $A^{(2)}_{02} = 1$, $A^{(2)}_{11} = 2$ and $A^{(2)}_{k,l} = 0$ otherwise.

Induction step: Let $(x+y)^{n-1} = \sum_{k,l} A^{(n-1)}_{k,l}\, x^k y^l$

$$(x+y)^n = (x+y)^{n-1}(x+y)$$

$$= \left(\sum_{k,l} A^{(n-1)}_{k,l}\, x^k y^l \right)(x+y)$$

$$= \sum_{k,l} A_{k,l}^{(n-1)} x^{k+1} y^{l} + \sum_{k,l} A_{k,l}^{(n-1)} x^{k} y^{l+1}$$

$$= \sum_{s,t} \left(A_{s-1,t}^{(n-1)} + A_{s,t-1}^{(n-1)} \right) x^{s} y^{t} .$$

So, the recursion and initial condition determining $A_{s,t}^{(n)}$ are

$$A_{n0}^{(n)} = 1 , \quad A_{0n}^{(n)} = 1 , \quad A_{s,t}^{(n)} = 0 \quad \text{unless } s+t=n .$$

and

$$A_{s,t}^{(n)} = A_{s-1,t}^{(n-1)} + A_{s,t-1}^{(n-1)} .$$

Yet, the condition that $A_{s,t}^{(n)} = 0$ unless $s+t=n$ allows us to write $A_{s,t}^{(n)} = A_s^{(n)}$ without confusion.

So, this recursion becomes:

$$A_s^{(n)} = A_{s-1}^{(n-1)} + A_s^{(n-1)}$$

$$\begin{tikzcd}[column sep=small]
 & & & A_0^{(0)} \arrow[dl, dash] \arrow[dr, dash] & & & \\
 & & A_0^{(1)} \arrow[dl, dash] \arrow[dr, dash] & & A_1^{(1)} \arrow[dl, dash] \arrow[dr, dash] & & \\
 & A_0^{(2)} \arrow[dl, dash] \arrow[dr, dash] & & A_1^{(2)} \arrow[dl, dash] \arrow[dr, dash] & & A_2^{(2)} \arrow[dl, dash] \arrow[dr, dash] & \\
A_0^{(3)} & & A_1^{(3)} & & A_2^{(3)} & & A_3^{(3)}
\end{tikzcd}$$

$$\vdots$$